MAS348 – 2013-14 Exam Solutions

1(i)(a)

- Players: $1, 2, \ldots, 2n+1$.
- Actions: each player has set of actions $\{A, B\}$.
- Payoff: Let V be the set of all sequences $(v_1, \ldots, v_{2n+1})$ where $v_1, \ldots, v_{2n+1} \in \{A, B\}$; each player i has utility function $u_i : V \to \{0, 1\}$ defined as follows. For each $v = (v_1, \ldots, v_{2n+1}) \in V$ write $A(v) = \#\{v_i \,|\, 1 \le i \le 2n+1,\ v_i = A\}$ and $B(v) = \#\{v_i \,|\, 1 \le i \le 2n+1,\ v_i = B\}$. If player i is a supporter of A, define $u_i(v) = 1$ if $A(v) > B(v)$ and $u_i(v) = 0$ otherwise. If player i is a supporter of B, define $u_i(v) = 0$ if $A(v) > B(v)$ and $u_i(v) = 1$ otherwise.

1(i)(b) Any strategy profile $v \in V$ with $|A(v) - B(v)| \ge 2$ is a Nash equilibrium, because changing the vote of any player will not change the outcome of the election.

If $A(v) = B(v) + 1$, v fails to be a Nash equilibrium only if there was a supporter of B who voted for A. If $B(v) = A(v) + 1$, v fails to be a Nash equilibrium only if there was a supporter of A who voted for B.

1(ii)(a) The profits for each company are given by $f_1 = (16 - 3q_1 - 4q_2)q_1$ and $f_2 = (24 - 4q_1 - 5q_2)q_2$. The best response $B_1(q_2)$ is obtained by solving $\partial f_1/\partial q_1 = 0$ yielding $B_1(q_2) = 8/3 - (2/3)q_2$. The best response $B_2(q_1)$ is obtained by solving $\partial f_2/\partial q_2 = 0$ yielding $B_2(q_1) = 12/5 - (2/5)q_1$.

1(ii)(b) A strategy profile (q_1, q_2) is a Nash equilibrium precisely when $q_1 = B_1(q_2), q_2 = B_2(q_1)$. We now solve the system of equations $q_1 = B_1(q_2), q_2 = B_2(q_1)$ to obtain $q_1 = 16/11$ and $q_2 = 20/11$.

1(iii) We eliminate dominated strategies sequentially as follows. We eliminate u and m as d dominates them and obtain

	L	M	R
d	3, 0	7, 4	8, 3

Now we can eliminate L and R as they both are dominated by M:

	M
d	7, 4

The solution of this game is (d, M).

(2)(i) Write $m = \max_{s\in S} \min_{t\in T} u(s,t)$. For all $s \in S$ and $t' \in T$ we have $u(s,t') \geq \min_{t\in T} u(s,t)$ so $\max_{s\in S} u(s,t') \geq \max_{s\in S} \min_{t\in T} u(s,t) = m$ so in particular $\min_{t\in T} \max_{s\in S} u(s,t) \geq m$. **(2)(ii)(a)** For any $p \in \Delta^R$ we have $p^T Ap = (p^T Ap)^T = p^T A^T p = -p^T Ap$, hence $p^T Ap = 0$ hence $V = \min_{y\in\Delta^C} \max_{x\in\Delta^R} x^T Ay \geq \min_{y\in\Delta^C} y^T Ay = 0$ and $V = \max_{x\in\Delta^R} \min_{y\in\Delta^C} x^T Ay \leq \max_{x\in\Delta^R} x^T Ax = 0$.

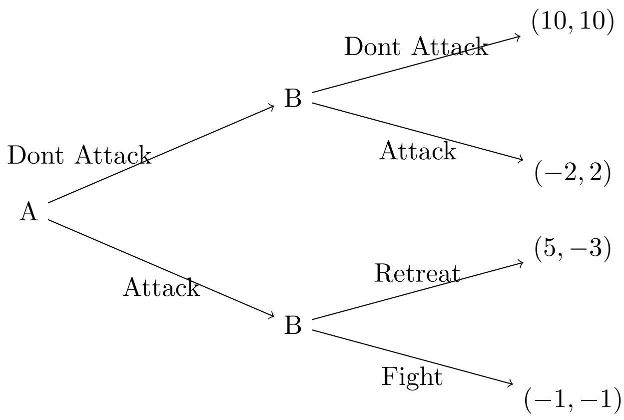
(2)(ii)(b) If $p \in \Delta_R$ is optimal for the row player, $p^T A$ has non-negative entries, and so $(p^T A)^T = A^T p = -Ap$ has non-negative entries and p is optimal for the column player.

(2)(iii) We look for a mixed strategy $p = (p_1, p_2, 1-p_1-p_2)$ such that (p,p) is an optimal strategy profile. The fact that the row player mixes all her pure strategies
imply that $-2p_2 + 2(1-p_1-p_2) = 2p_1 - 3(1-p_1-p_2) = -2p_1 + 3p_2$ The solution of this system of equations is $p_1 = 3/7$, $p_2 = 2/7$, hence $((3/7, 2/7, 2/7), (3/7, 2/7, 2/7))$ is a symmetric optimal strategy pair.

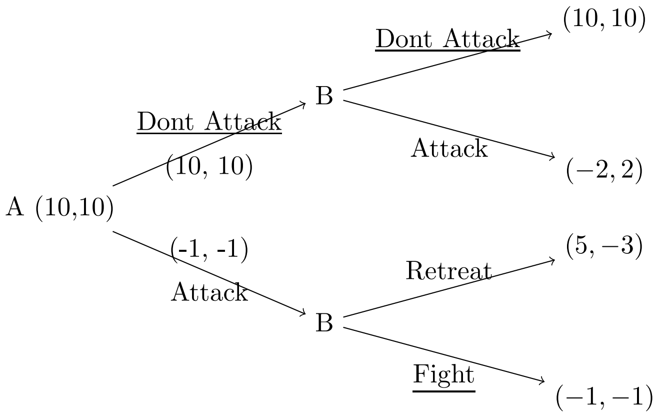
(2)(iv) The following game is a counter-example:

	A	B
I	0	0
II	0	0

(3)(i)(a)



(3)(i)(b) A simple backward induction yields



hence the only subgame perfect Nash equilibrium is the stragegy instructing A not to attack, B to fight if attacked and B to not attack if not attacked.

(3)(i)(c) The strategic form of the game is given in tabular for as follows:

	[Fight, Attack]	[Fight, Dont Attack]	[Retreat, Attack]	[Retreat, Dont Attack]
Attack	-1, -1	-1, -1	5, -3	5, -3
Don't Attack	-2, 2	10, 10	-2, 2	10, 10

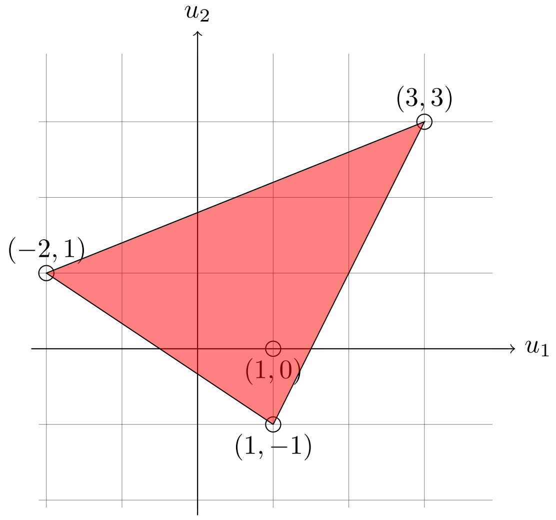
(3)(i)(d) We indicate A and B's best responses as follows

	[Fight, Attack]	[Fight, Dont Attack]	[Retreat, Attack]	[Retreat, Dont Attack]
Attack	[-1], [-1]	-1, [-1]	[5], -3	5, -3
Don't Attack	-2, 2	[10], [10]	-2, 2	[10], [10]

and we identify 3 Nash Equilibria: (Don't Attack, [Fight, Don't Attack]),
(Don't Attack, [Retreat, Don't Attack]), and
(Attack, [Fight, Attack]). The first corresponds to the subgame perfect Nash equilibrium found in (c), while the last two are Nash equilibria which is not subgame perfect.

(3)(ii) Zermelo's Theorem implies that if white does not have a strategy which guarantees victory or a draw, then black has a strategy which guarantees victory to him. Assume this and use a strategy stealing argument: white will initially pass turning the game into one in which black goes first, or resulting in a draw if Black passes. Subsequently white will adopt black's strategy in the original game. This gives White a winning strategy, contradicting our assumption.

(4)(i)(a) The *minimax values* of players 1 and 2 are $\min_{t\in T}\max_{s\in S} u_1(s,t)$ and $\min_{s\in S}\max_{t\in T} u_2(s,t)$. respectively. (Thus the minimax value of a player is the worst possible payoff the other player can inflict.)
(4)(i)(b) The *cooperative payoff region* of G is the convex hull of $\{(u_1(s,t), u_2(s,t) \,|\, s\in S, s\in T\} \subseteq \mathbb{R}^2$.
(4)(i)(c) The minimax values are $m_1 = 1$ and $m_2 = 0$ and the cooperative payoff region is



(4)(i)(d) We solve $(2,2) = a(-2,1) + b(3,3) + c(1,-1)$ together with the condition $a+b+c = 1$ and obtain $a = 2/16, b = 11/16, c = 3/16$. *[Many other solutions exist, e.g.,* $(6/9)\times(3,3) + (1/9)\times(1,-1) + (1/9)\times(1,0) + (1/9)\times(-2,1)$*]*
(4)(i)(e) Consider the row player strategy s_1 consisting of playing repeatedly the pattern: 13 times II followed by 3 times I. Consider the column player strategy s_2 consisting of playing repeatedly the pattern: 2 times B followed by 14 times A.

The payoff resulting from playing the strategy pair (s_1, s_2) is $(2,2)$ (but this is not a Nash equilibrium, necessarily).

The Nash equilibrium resulting in the desired payoff consists of the row player playing s_1 if the other player has played s_2 thus far, and playing I otherwise

and the column player playing s_2 if the other player has played s_1 thus far, and playing B otherwise.
(4)(ii)(a) Notice first that Alice should not bid less than 0 or more than £11,000. The set of actions for Alice is the set of integers in $[0,11]$, and Bob has two actions available: Accept and Reject. Alice has one type: Alice; Bob has 10 types $B_1,\ldots,B_{10}$ where Bob has type B_i when the value of his diamond is $\pounds i\times 1000$. Ω is the set of integers in $[1,10]$, $\tau_A(i) = \text{Alice}$, $\tau_B(i) = B_i$. Alice has 12 strategies: her 12 possible bids. Bob's strategies is the set of functions from $\{B_1,\ldots,B_{10}\}$ to $\{\text{Accept},\text{Reject}\}$. $\text{P}(i|\text{Alice}) = 1/10$; $\text{P}(i|B_j)$ is 0 if $i \neq j$ and 1 if $i = j$.
(4)(ii)(b) B_i accepts a bid of $k\times 1000$ iff $k > i$, so Alice's strategy of bidding k has expected payoff

$$1000\sum_{i=1}^{k-1}\frac{1}{10}\times(3\times i-k) = 50\left(k^2-k\right)$$

which attains a maximum at $k = 11$, so the only Bayes-Nash equilibrium occurs when Alice bids 11,000 and all types of Bob accept it.