The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2013–2014

Game Theory — **2 hours and 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Two candidates A and B run for office in an election where an odd number of voters must vote for one or the other (abstentions are not allowed). Each voter is a supporter of exactly one of the candidates, and they assign higher utility to the victory of their candidate than to the victory of the other candidate.

(a) Describe this situation as a game in strategic form. ***(6 marks)***

(b) Find all pure-strategy Nash-equilibria of this game. ***(5 marks)***

(ii) Suppose that two firms produce similar but not identical products, and that the unit costs of these products are £4 for firm 1 and £6 for firm 2. The prices of each of these two products depend on the production profile (q_1, q_2) of both products: $p_1 = 20 - 3q_1 - 4q_2$ and $p_2 = 30 - 4q_1 - 5q_2$. Assume that each firm i controls its production profile q_i.

(a) Find each company's production which is the best response to the other company's production. ***(6 marks)***

(b) Find the production profile which is a Nash-equilibrium. ***(4 marks)***

(iii) Alice and Bob play a game given in strategic form as follows:

	L	M	R
u	0, 1	1, 5	2, 2
m	2, 5	5, 4	4, 9
d	3, 0	7, 4	8, 3

Solve this game. ***(4 marks)***

2 (i) Consider a finite, two-player, zero-sum game (S, T, u). Show that

$$\min_{t \in T} \max_{s \in S} u(s,t) \geq \max_{s \in S} \min_{t \in T} u(s,t).$$

(5 marks)

(ii) A finite zero-sum game $G = (S, T, u)$ is *symmetric* if $S = T$ and for all $s_1, s_2 \in S$, $u(s_2, s_1) = -u(s_1, s_2)$. Let $A = (u(i,j))$ be the matrix associated with a symmetric game $G = (\{1,\ldots,n\}, \{1,\ldots,n\}, u)$.

(a) Show that the value V of a symmetric game is zero. ***(5 marks)***

(b) Show that an optimal strategy for one player is also an optimal strategy for the other player. ***(5 marks)***

(iii) Consider the following zero-sum game given in tabular form

	A	B	C
I	0	-2	2
II	2	0	-3
III	-2	3	0

Find an optimal mixed-strategy profile for this game where each strategy includes all pure strategies with positive probability. ***(8 marks)***

(iv) Must all optimal mixed-strategy profiles of finite, symmetric, zero-sum games be of the form (p, p)? Justify your answer. ***(2 marks)***

3 (i) Country A will either attack country B or not attack it. If A attacks, B can fight, resulting in payoffs of -1 for both, or retreat, resulting in a payoff of 5 to A and -3 to B. If A does not attack, B will either attack, resulting in a payoff of 2 to B and -2 to A, or B will not attack, resulting in a payoff of 10 to both.

(a) Describe this game using a tree, carefully labelling all its components. ***(5 marks)***

(b) Find all subgame perfect Nash-equilibria of the game. ***(5 marks)***

(c) Describe the game in strategic form. ***(5 marks)***

(d) Find a pure-strategy Nash-equilibrium of this game which is not subgame perfect. ***(5 marks)***

(ii) Consider a game identical to chess, except that

- each player may choose to pass and not make a move when it is their turn, and
- the game ends with a draw after two consecutive passes.

Prove that white (who is the first player to move) has a strategy that guarantees victory or a draw. ***(5 marks)***

4 (i) Consider a 2-player game given in strategic form as (S, T, u_1, u_2).

(a) Define the *minimax values* of both players. ***(2 marks)***

(b) Define the *cooperative payoff region* of the game. ***(2 marks)***

(c) Find the minimax values and sketch the cooperative payoff region of the following 2-person game G given in tabular form as follows

	A	B
I	1, -1	1, 0
II	3, 3	-2, 1

(4 marks)

(d) Show that the point $(2, 2)$ is in the cooperative payoff region of G by writing it as a convex combination of payoffs. ***(2 marks)***

(e) Consider now the game G^∞ which consists of playing G repeatedly, and where the payoffs of the infinite game are the average payoffs of the individual matches. Describe, without proof, a Nash-equilibrium which results in average payoff of 2 for both players. ***(6 marks)***

(ii) Alice wants to buy a diamond from Bob. The value of the diamond is £$1000k$ for an integer $1 \leq k \leq 10$; this value is known to Bob but not to Alice, and Alice assumes that all values are equally probable. Alice, who is a talented jeweller, knows that her labour can triple the value of the diamond. Bob asks Alice to submit a bid of an integer multiple of £1000 for the diamond. Assume that Bob accepts a bid if and only if it is strictly higher than the value of the diamond, and assume that Alice knows this.

(a) Model this as a Bayesian game. ***(5 marks)***

(b) Find all Bayes-Nash-equilibria of this game. ***(4 marks)***

End of Question Paper