MAS348 – 2014-15 Exam Solutions

Question 1
unseen, similar to homework and class examples.

1(i)(a)

- Players: $P_1, P_2, \ldots, P_{20}$.
- Actions: each player has set of actions $\{1, 2, \ldots, 10\}$.
- Payoff: Let $v = (v_1, \ldots, v_{20})$ list the choices of $P_1, \ldots, P_{20}$. Then

$$u_i(v) = \begin{cases} 1, & \text{if } v_i \text{ was chosen strictly more frequently than any other number} \\ 0, & \text{otherwise.} \end{cases}$$

1(i)(b) The only Nash equilibria of this game consist of all people choosing the same integer. When such a strategy profile is played, any deviation will change a payoff of £1 to £0, hence it is a Nash equilibrium. To see that there are no other Nash equilibria note that

- a player who chose a number with frequency less than the maximal would benefit by switching to a number with maximal frequency, and
- if two numbers are chosen with maximal frequency, players who chose these would benefit from switching to the other number chosen with maximal frequency.

1(ii)(a) The profit for company 1 given a production profile (q_1, q_2) is
$f(q_1, q_2) = (6000 - 2(q_1 + q_2) - q_1)q_1$ and to find the best response to company 2 production of q_2 we find the value q_1 which maximizes $f(q_1, q_2)$ for the given q_2, e.g., by solving $\partial f(q_1, q_2)/\partial q_1 = 0$ and we obtain $q_1 = 1000 - q_2/3$, hence company 1 has a best response function $B_1(q_2) = 1000 - q_2/3$. The symmetry of the situation gives a best response function $B_2(q_1) = 1000 - q_1/3$ for company 2.
1(ii)(b) A strategy profile (q_1, q_2) is a Nash equilibrium precisely when $q_1 = B_1(q_2), q_2 = B_2(q_1)$. We now solve this system of equations to obtain $q_1 = q_2 = 750$.
1(iii) Both Alice's and Bob's choice of Disneyland is dominated, so we can delete these without losing any Nash equilibria, and we obtain a simplified game

	seaside	ski
seaside	5,4	2,1
ski	1,2	4,5

Both strategy profiles (seaside, seaside) and (ski, ski) are Nash equilibria.

We now find mixed strategy Nash equilibria of the form $((p, 1-p), (q, 1-q))$. The principle of indifference gives $4p + 2(1-p) = p + 5(1-p)$ and $5q + 2(1-q) = q + 4(1-q)$ and we obtain $p = 1/2$ and $q = 1/3$.

Question 2

(i) is bookwork, (ii) is unseen, similar to homework problems and class examples.

(2)(i)(a) Write $S = \{s_1, \ldots, s_m\}$ and $T = \{t_1, \ldots, t_n\}$. Then Δ^R consists of all probability vectors $\{(p_1, \ldots, p_m)^T \,|\, 0 \le p_1, \ldots, p_m \le 1, p_1 + \cdots + p_m = 1\}$ and Δ^C consists of all probability vectors $\{(q_1, \ldots, q_n)^T \,|\, 0 \le q_1, \ldots, q_n \le 1, q_1 + \cdots + q_n = 1\}$.

(2)(i)(b) The value of the game V is the common value of
$\min_{q \in \Delta^C} \max_{p \in \Delta^R} u(p, q) = \max_{p \in \Delta^R} \min_{q \in \Delta^C} u(p, q)$

(2)(i)(c) A strategy p^* for the row player is optimal if $\max_{p \in \Delta^R} \min_{q \in \Delta^C} u(p, q) = \min_{q \in \Delta^C} u(p^*, q)$, and a strategy q^* for the column player is optimal if $\min_{q \in \Delta^C} \max_{p \in \Delta^R} u(p, q) = \max_{p \in \Delta^R} u(p, q^*)$.

(2)(i)(d) Choose any $p \in \Delta^R$ and $q \in \Delta^C$.

$$u(p^*, q) \ge \min_{q \in \Delta^C} u(p^*, q) = \max_{p \in \Delta^R} \min_{q \in \Delta^C} u(p, q) = \min_{q \in \Delta^C} \max_{p \in \Delta^R} u(p, q) = \max_{p \in \Delta^R} u(p, q^*) \ge u(p, q^*)$$

and in particular $u(p^*, q) \ge u(p^*, q^*)$ and $u(p, q^*) \le u(p^*, q^*)$, hence (p^*, q^*) is a Nash equilibrium.

(2)(ii)(a) A saddle point (s, t) would correspond to a pure-strategy Nash equilibrium. We look for these by indicating best responses for both players as follows: the best responses of the row player to each of the column player's pure strategies are

	A	B	C
I	<u>1</u>	-1	<u>2</u>
II	-2	0	1
III	0	<u>2</u>	-1

while the best responses of the column player to each of the row player's pure strategies are

	A	B	C
I	1	<u>-1</u>	2
II	<u>-2</u>	0	1
III	0	2	<u>-1</u>

and we see that there is no pure-strategy Nash equilibrium hence the game has no saddle point.

(2)(ii)(b) Define

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

We compute ${p^*}^T A = [1/2, 1/2, 1/2]$ and hence any mixed strategy of the column player is a best response. Since by playing p^* the row player guarantees a payoff of $1/2$, we deduce that $V \ge 1/2$.

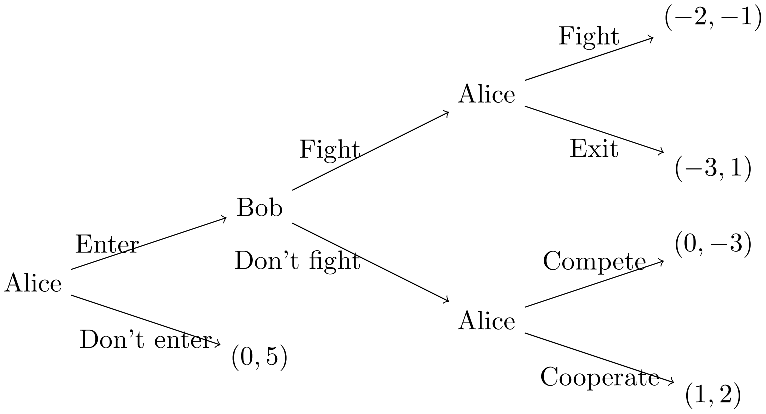
(2)(ii)(c) We compute $Aq^* = [1/2, -3/2, 1/2]^T$ and hence the row player best responses are all mixed strategies not mixing II. Since by playing q^* the column player is guaranteed not to lose more than $1/2$, we deduce that $V \le 1/2$, hence $V = 1/2$.

(2)(ii)(d) Let (x, y) be a mixed strategy profile consisting of optimal strategies and assume that $x(II) > 0$. Now $1/2 = u(x, y) \le u(x, q^*) = x(I)/2 + x(III)/2 - 3x(II)/2 = (1/2)(x(I) + x(III) - 3x(II)) < 1/2$, a contradiction.

Question 3

(i) is unseen, similar to homework and class examples, (ii) was a homework problem.

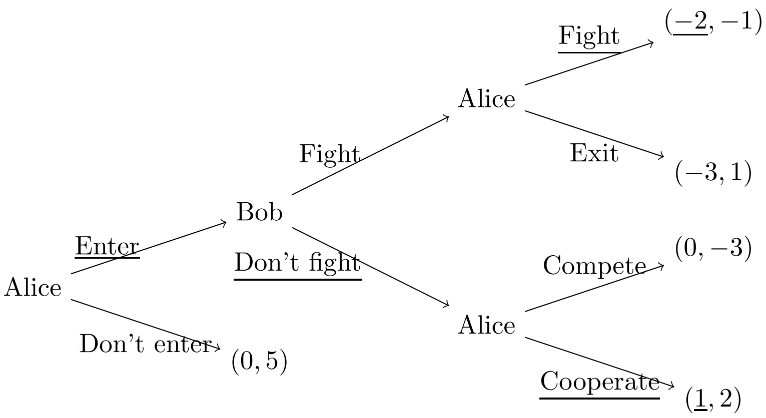
(3)(i)(a)



where payoffs are in millions of pounds.

[leaves] [edges] [other nodes]

(3)(i)(b) A simple backward induction yields



If Alice enters then she prefers to fight a fighting Bob ($-2 > -3$) and she prefers to cooperate with a non-fighting Bob ($1 > 0$), hence Bob chooses to not fight for a payoff of 2 rather than fight for a payoff of -1, thus ensuring Alice a payoff if she enters which exceeds her 0 if she doesn't. Hence Alice will enter the market.

(3)(i)(c) The strategic form of the game is given in tabular for as follows:

	Fight	Don't fight
[Enter, Fight, Cooperate]	-2, -1	1, 2
[Enter, Fight, Compete]	-2, -1	0, -3
[Enter, Exit, Cooperate]	-3, 1	1, 2
[Enter, Exit, Compete]	-3, 1	0, -3
[Don't enter, Fight, Cooperate]	0, 5	0, 5
[Don't enter, Fight, Compete]	0, 5	0, 5
[Don't enter, Exit, Cooperate]	0, 5	0, 5
[Don't enter, Exit, Compete]	0, 5	0, 5

(3)(i)(d) We indicate Alice and Bob's best responses as follows

	Fight	Don't fight]
[Enter, Fight, Cooperate]	-2,-1	1, 2
[Enter, Fight, Compete]	-2, -1	0, -3
[Enter, Exit, Cooperate]	-3, 1	1, 2
[Enter, Exit, Compete]	-3,1	0, -3
[Don't enter, Fight, Cooperate]	0,5	0,5
[Don't enter, Fight, Compete]	0,5	0,5
[Don't enter, Exit, Cooperate]	0,5	0,5
[Don't enter, Exit, Compete]	0,5	0,5

and we identify 6 Nash Equilibria, 5 not being subgame perfect, e.g., ([Enter, Exit, Cooperate], Don't fight).

(3)(ii) Assume this is not the case; hence Zermelo's Theorem implies that the second player has a strategy which guarantees him victory. The first player now steals that strategy as follows.

The first player starts by picking any random square S and henceforth plays the game as if she were the second player and as if S were empty. If, later in the game, the strategy calls for taking a square already owned by the first player, she makes any random game. The game ends either with a winning position in the second player's winning strategy book (with players interchanged), or with such a wining position plus an extra square for the first player, which is also a winning position.

Question 4

unseen, similar to homework and class examples.

(4)(i)(a) The expected payoff for both players is $\sum_{i=0}^{\infty} 3p^i = 3/(1-p)$.

(4)(i)(b) Bob's best response to Alice's II is B, yielding a payoff of 1 in each repetition of the game. Thus Bob's expected payoff is $\sum_{i=0}^{\infty} p^i = 1/(1-p)$.

(4)(i)(c) This is not a Nash equilibrium because if Alice changes her strategy to playing II at every stage

she would increase her expected payoff from $3/(1-p)$ to $\sum_{i=0}^{\infty} 5p^i = 5/(1-p)$.

(4)(i)(d) The strategy profile (II, B) is a Nash equilibrium of G, hence it is a Nash equilibrium for the repeated game.

(4)(i)(e) Consider Alice's strategy $\mathcal{A}$ consisting of playing I as long as Bob did not ever play B, and playing II forever once Bob played B. Consider Bob's strategy $\mathcal{B}$ consisting of playing A as long as Alice did not ever play II, and playing B forever once Alice played II. The strategy profile ($\mathcal{A}$, $\mathcal{B}$) results in an expected payoff of $3/(1-p)$. If either player deviates at the nth repetition of this game, their expected payoff will be at most $\sum_{i=0}^{n-2} 3p^i + 5p^{n-1} + \sum_{i=n}^{\infty} p^i$ and for the deviation not to pay off we must have $\sum_{i=0}^{n-2} 3p^i + 5p^{n-1} + \sum_{i=n}^{\infty} p^i \leq \sum_{i=0}^{\infty} 3p^i$, i.e. $5p^{n-1} + \sum_{i=n}^{\infty} p^i \leq \sum_{i=n-1}^{\infty} 3p^i$, i.e. $5p^{n-1} + p^n/(1-p) \leq 3p^{n-1}/(1-p)$, which yields $2p^{n-1}(1-2p) \leq 0$ and hence $p \geq 1/2$. We take $p_0 = 1/2$.

(4)(ii)(a) Alice has two types, clever and dull, hence each of her strategies strategy needs to specify her actions for each of her types. The set of her strategies is { [Accept, Accept], [Accept, Decline], [Decline, Accept], [Decline, Decline] }. Bob has only one type and the set of his strategies coincides with the set of his actions.

(4)(ii)(b) We compile the expected payoffs of all strategy profiles as follows

	Accept	Decline
[Accept, Accept]	[10, 2], 2	[-2, -2], 0
[Accept, Decline]	[10, -1], 6	[-2, 0], 0
[Decline, Accept]	[0, 2], -26/5	[0, -2], 0
[Decline, Decline]	[0, -1], -6/5	[0, 0], 0

and find best responses

	Accept	Decline
[Accept, Accept]	[$\underline{10}$, $\underline{2}$], $\underline{2}$	[-2, -2], 0
[Accept, Decline]	[$\underline{10}$, -1], $\underline{6}$	[-2, $\underline{0}$], 0
[Decline, Accept]	[0, $\underline{2}$], -26/5	[$\underline{0}$, -2], $\underline{0}$
[Decline, Decline]	[0, -1], -6/5	[$\underline{0}$, $\underline{0}$], $\underline{0}$

hence there are two Bayes-Nash equilibria: all types of Alice accept and Bob accepts, all types of Alice decline and Bob declines.