MAS348 – 2015-16 Solutions

Question 1
unseen, similar to homework and class examples.

1(i)(a)

- A set of Players: $\{P_1, P_2, \dots, P_{20}\}$. **(+1)**

- Actions: each player has set of actions $\{\text{plot } 1, \text{plot } 2, \dots, \text{plot } 10\}$. **(+1)**

- Payoff: Let $v = (v_1, \dots, v_{20})$ list the choices of $P_1, \dots, P_{20}$. Then $u_i(v) = 100000/\#\{j \mid v_j = v_i\}$. **(+2)**

1(i)(b) If all plots are chosen by exactly 2 people, no one would choose to move and share a plot with 3 people, hence this a Nash equilibrium. **(+2)**
If some plot has more than 2 people, another must have at most 1, and those in the crowded plot would regret not choosing the plot with at most one person. **(+3)**

1(i)(c) Everyone choosing any of the plots with probability 1/10 is a Nash equilibrium. **(+5)**

1(ii)(a)

- A set of Players: $\{\text{Alice}, \text{Bob}\}$. **(+1)**

- Actions: each player has set of actions $[0, \infty) \subset \mathbb{R}$. **(+1)**

- Payoff: when Alice plays a and Bob b, Alice's payoff is $A(a,b) = 8\sqrt{a+b} - a$ and Bob's payoff is $B(a,b) = 8\sqrt{a+b} - b$. **(+1)**

1(ii)(b) Alice's best response function is obtained by solving $\frac{\partial A}{\partial a} = 4/\sqrt{a+b} - 1 = 0$ giving $BR_A(b) = 16 - b$ and similarly $\frac{\partial B}{\partial b} = 4/\sqrt{a+b} - 1 = 0$ yields $BR_B(a) = 16 - a$. **(+2)**

A strategy profile (a^*, b^*) is a Nash equilibrium precisely when $a^* = BR_A(b^*), b^* = BR_B(a^*)$. This is satisfied whenever $a^* + b^* = 16$ and each of these gives a Nash equilibrium. **(+2)**

1(ii)(c) If both commit to work h hours they each get a payoff of $P(h) = 8\sqrt{2h} - h$ and this is maximized for $h = 32$, yielding both a payoff of 32. **(+2)**
Since $h + h \neq 16$, each working 32 hours is not a Nash equilibrium, **(+1)**
and in the absence of the legally binding agreement they would deviate from working 32 hours. **(+1)**

Question 2

(i) is bookwork, (ii) is unseen, similar to homework problems and class examples.

(2)(i)(a) Δ^R consists of all probability vectors $\{(p_1,\dots,p_m)^T \mid 0 \le p_1,\dots,p_m \le 1, p_1+\cdots+p_m = 1\}$ and Δ^C consists of all probability vectors $\{(q_1,\dots,q_n)^T \mid 0 \le q_1,\dots,q_n \le 1, q_1+\cdots+q_n = 1\}$. **(+2)**

(2)(i)(b) $p^t M q$ **(+1)**

(2)(i)(c) The value of the game V is the common value of
$\min_{q\in\Delta^C} \max_{p\in\Delta^R} u(p,q) = \max_{p\in\Delta^R} \min_{q\in\Delta^C} u(p,q)$ **(+2)**

(2)(i)(d) A strategy p^* for the row player is optimal if $\max_{p\in\Delta^R} \min_{q\in\Delta^C} u(p,q) = \min_{q\in\Delta^C} u(p^*,q)$, **(+1)**
and a strategy q^* for the column player is optimal if $\min_{q\in\Delta^C} \max_{p\in\Delta^R} u(p,q) = \max_{p\in\Delta^R} u(p,q^*)$. **(+1)**

(2)(i)(e) Let V be value of the game.

$$V = \max_{p\in\Delta_m} \min_{q\in\Delta_n} p^t M q \le \max_{p\in\Delta_m} p^t M q^* \le p^t \begin{bmatrix} v \\ \vdots \\ v \end{bmatrix} = v$$

(+2)

$$V = \min_{q\in\Delta_n} \max_{p\in\Delta_m} p^t M q \ge \min_{q\in\Delta_n} {p^*}^t M q \ge (v,\dots,v) q = v$$

hence $v = V$. **(+2)**

Now $V = \max_{p\in\Delta^R} \min_{q\in\Delta^C} u(p,q) \ge \min_{q\in\Delta^C} u(p^*,q) \ge V$, hence $\min_{q\in\Delta^C} u(p^*,q) = V$ and p^* is optimal. Also $V = \min_{q\in\Delta^C} \max_{p\in\Delta^R} u(p,q) \le \max_{p\in\Delta^R} u(p,q^*) \le V$ hence $\max_{p\in\Delta^R} u(p,q^*) = V$ and q^* is optimal. **(+2)**

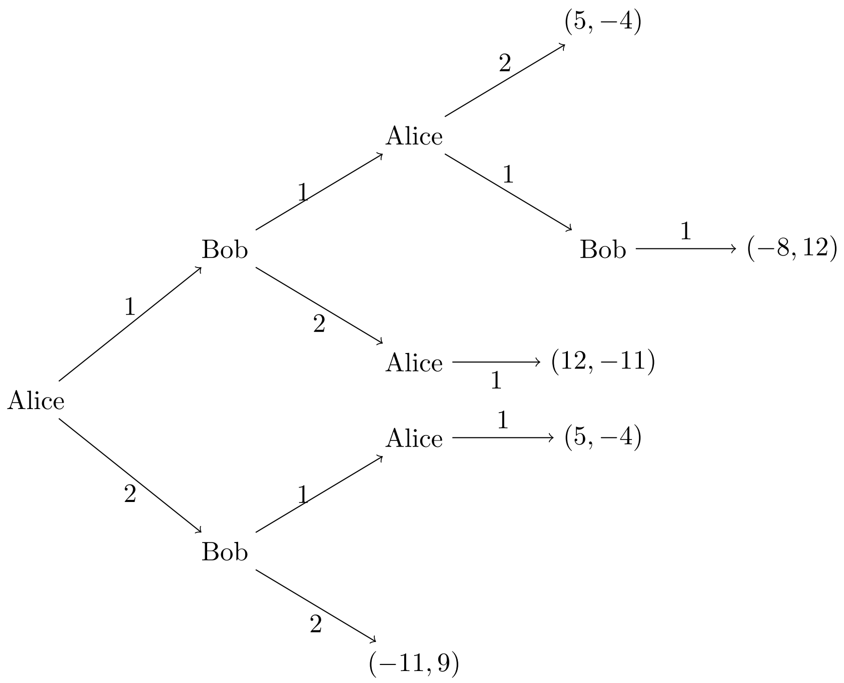
(2)(ii) Write $M = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -5 & 2 & 0 \end{bmatrix}$ and consider mixed-strategies $p = (p_1, p_2, 1-p_1-p_2)^t$ and $q = (q_1, q_2, 1-q_1-q_2)^t$ for Alice and Bob. **(+2)**
The principle of indifference implies that for (p,q) to be a Nash equilibrium we must satisfy $-p_2 + 5(1-p_1-p_2) = p_1 - 2(1-p_1-p_2) = -2p_1 + p_2$ **(+3)**
and $-q_2 + 2(1-q_1-q_2) = q_1 - (1-q_1-q_2) = -5q_1 + 2q_2$ **(+3)**
giving $p = (9/32, 19/32, 1/8)^t$ and $q^t = (7/32, 17/32, 1/4)$. **(+2)**

We compute $p^t M = [-1/32, -1/32, -1/32]$ and $Mq = \begin{bmatrix} -1/32 \\ -1/32 \\ -1/32 \end{bmatrix}$ and using part (i)(d) we deduce that p and q are optimal strategies, and that the value of the game is $V = -1/32$. **(+2)**

Question 3

(i) is unseen, similar to homework and class examples, (ii) was a homework problem.

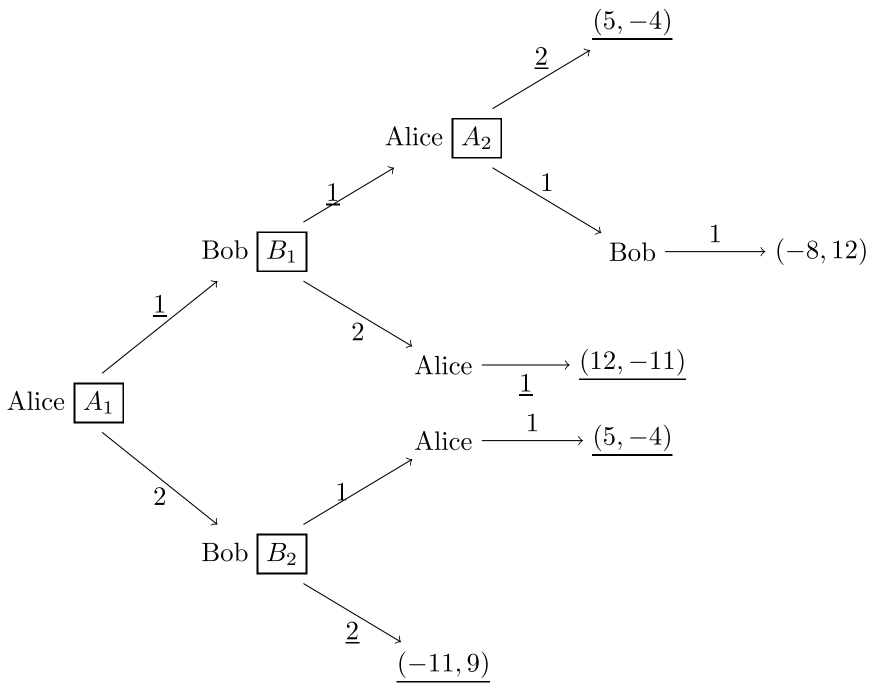
(3)(i)(a)



where payoffs are in millions of pounds.

[leaves] **(+2)** [edges] **(+3)** [other nodes] **(+2)**

(3)(i)(b) A simple backward induction yields



Thus Alice will get the patent, she will earn 5 million pounds and Bob will lose 4 millions. **(+5)**

(3)(i)(c) Alice's strategies will be denoted $[ij]$ where i and j are the number of steps decided in vertices A_1 and A_2. Bob's strategies will be denoted $[ij]$ where i and j are the number of steps decided in vertices B_1 and B_2. The strategic form of the game is given in tabular form as follows:

	[11]	[12]	[21]	[22]
[11]	(-8,12)	(-8,12)	(12,-11)	(12, -11)
[12]	(5,-4)	(5,-4)	(12,-11)	(12, -11)
[21]	(5,-4)	(-11, 9)	(5,-4)	(-11, 9)
[22]	(5,-4)	(-11, 9)	(5,-4)	(-11, 9)

[strategies] **(+2)**
[payoffs] **(+3)**

We indicate Alice and Bob's best responses as follows

	[11]	[12]	[21]	[22]
[11]	(-8,$\underline{12}$)	(-8,$\underline{12}$)	($\underline{12}$,-11)	($\underline{12}$, -11)
[12]	($\underline{5}$,$\underline{-4}$)	($\underline{5}$,$\underline{-4}$)	($\underline{12}$,-11)	($\underline{12}$, -11)
[21]	($\underline{5}$,-4)	(-11, $\underline{9}$)	(5,-4)	(-11, $\underline{9}$)
[22]	($\underline{5}$,-4)	(-11, $\underline{9}$)	(5,-4)	(-11, $\underline{9}$)

and we identify 2 Nash Equilibria, **(+2)**
$([12, 12])$ is subgame-perfect $([12, 11])$ is not subgame-perfect **(+1)**

(3)(ii) We use the following *strategy-stealing argument.* Assume that the Theorem fails; Zermelo's theorem implies that the second player has a strategy which guarantees him victory. **(+2)**
If the first player chooses the upper-right piece, there is a choice of piece P for the second player which is the first step in a winning strategy. Let the first player then start with a choice of P, and let him follow the winning strategy with the roles reversed, as if the first player moved second. **(+3)**

Question 4

unseen, similar to homework and class examples.

(4)(i)(a) The *minimax values* of players 1 and 2 are $\min_{t\in T}\max_{s\in S} u_1(s,t)$ and $\min_{s\in S}\max_{t\in T} u_2(s,t)$. respectively. **(+2)**

(Thus the minimax value of a player is the worst possible payoff the other player can inflict.)

(4)(i)(b) The *cooperative payoff region* of G is the convex hull of $\{(u_1(s,t), u_2(s,t)) \mid s \in S, t \in T\} \subseteq \mathbb{R}^2$. **(+2)**

(4)(ii)(a) Notice that when G is played once II dominates I and after eliminating I, A dominates B. When the game is played in a sequence of known length, the first player will play II and the second A. **(+2)**

(4)(ii)(b) Following the hint we look for $\alpha, \beta \in [0,1]$ such that $(3,3) = \alpha(0,2) + \beta(2,0) + (1-\alpha-\beta)(4,5)$ and this gives $\alpha = 1/14$, $\beta = 5/14$. **(+2)**

(4)(ii)(c) The minimax values are $m_1 = 2$ and $m_2 = 2$ corresponding to the column player playing B and the row player playing I, respectively. **(+1)**

Consider the row player strategy s_1 consisting of playing repeatedly the pattern: 1 time I followed by 5 times II and followed by 8 times II. **(+2)**

Consider the column player strategy s_2 consisting of playing repeatedly the pattern: 1 times B followed by 5 times B followed by 8 times A. **(+2)**

The payoff resulting from playing the strategy pair (s_1, s_2) is $(3,3)$ (but this is not a Nash equilibrium yet).

The Nash equilibrium resulting in the desired payoff consists of the row player playing s_1 if the other player has played s_2 thus far, and playing I otherwise and the column player playing s_2 if the other player has played s_1 thus far, and playing B otherwise. **(+1)**

(4)(iii)(a) Both Alice and Bob have one type. Alice's set of strategies is $\{U, D\}$ and Bob's is $\{L, R\}$. **(+1)**

We compile the expected payoffs of all strategy profiles as follows

	L	R
U	3,3	1,-1
D	-1, 1	2, 2

(+2)

and find best responses

	L	R
U	$\underline{3}$,$\underline{3}$	1,-1
D	-1, 1	$\underline{2}$, $\underline{2}$

(+2)

uncovering the Nash equilibria (U, L) and (D, R).

(4)(iii)(b) Now Alice has two types, A_1 who knows she is playing game 1, and A_2 who knows she's playing game 2, hence each of her strategies needs to specify her actions for each of her types. The set of her strategies is { [U, U], [U, D], [D, U], [D, D] }. Bob has only one type and the set of his strategies coincides with the set of his actions. **(+1)**

We compile the expected payoffs of all strategy profiles as follows

	L	R
$[UU]$	[1,9],3	[3, -5], -1
$[UD]$	[1,-10], 13/4	[3, 5], 20/4
$[DU]$	[2, 9], 3/4	[1, -5], -4
$[DD]$	[2,-10], 1	[1, 5], 2

(+3)

and find best responses

	L	R
$[UU]$	[1,$\underline{9}$],$\underline{3}$	[$\underline{3}$, -5], -1
$[UD]$	[1,-10], 13/4	[$\underline{3}$, $\underline{5}$], $\underline{20/4}$
$[DU]$	[$\underline{2}$, $\underline{9}$], $\underline{3/4}$	[1, -5], -4
$[DD]$	[$\underline{2}$,-10], 1	[1, $\underline{5}$], $\underline{2}$

(+2)

hence there are two Bayes-Nash equilibria: ([U D], R) and ([D U], L).