The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2015–2016

Game Theory — **2 hours and 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) A fertile valley is divided into ten plots of land, each yielding exactly the same amount of agricultural output worth £100,000. Twenty people settle in the valley, and each must choose one of the plots to live off. If more than one person chooses the same plot, its yield is shared equally among its occupants. If a plot is not chosen by anyone, its harvest is lost.

(a) Describe this situation as a game in strategic form. ***(4 marks)***

(b) Find all pure-strategy Nash equilibria of this game. Explain your answer in detail. ***(5 marks)***

(c) Describe, without justification, one mixed-strategy Nash equilibrium of this game. ***(5 marks)***

(ii) Alice and Bob collaborate on a project, each devoting some number of hours to it. If Alice works $x \geq 0$ hours and Bob works $y \geq 0$ hours, their combined wages will be $£16\sqrt{x+y}$, which they share equally. For both, the cost of devoting h hours to the project is $£h$.

(a) Describe this situation as a game in strategic form. ***(3 marks)***

(b) Find all pure-strategy Nash equilibria of this game. ***(4 marks)***

(c) Alice and Bob are offered the chance to enter a legally binding agreement, before the project starts, specifying an equal number of hours each will devote to the project. Once they enter this agreement, the penalty for not following through is so high that we can assume they will honour it. What is the most advantageous agreement for both Alice and Bob? Would they stick to it in the absence of the legally binding agreement? ***(4 marks)***

2 (i) Let $G = (\{s_1, \dots, s_m\}, \{t_1, \dots, t_n\}, u)$ be a zero-sum game. Let M be the $m \times n$ matrix with $M_{ij} = u(s_i, t_j)$.

(a) Describe the sets Δ^R and Δ^C of mixed strategies of both players.
(2 marks)

(b) If Alice plays $p \in \Delta^R$ and Bob plays $q \in \Delta^C$, what is the expected payoff $u(p, q)$ for Alice? ***(1 mark)***

(c) Define the *value* of the game. ***(2 marks)***

(d) What is meant by an *optimal strategy* of G? ***(2 marks)***

(e) Suppose that $p^* \in \Delta^R$ and $q^* \in \Delta^C$ are such that the minimal coordinate in ${p^*}^t M$ and the maximal coordinate in Mq^* both equal v. Prove that the value of the game is v, that p^* is an optimal strategy for the row player, and that q^* is an optimal strategy for the column player. ***(6 marks)***

(ii) Consider the following zero-sum game given in tabular form

	A	B	C
I	0	-1	2
II	1	0	-1
III	-5	2	0

and let V be the value of the game.

Identify a possible Nash equilibrium (p^*, q^*) of this game in which every strategy mixes all actions with positive probability, and verify that it is indeed a Nash equilibrium.
(12 marks)

3 (i) Alice and Bob own competing pharmaceutical companies trying to develop a cure for the common cold. Both companies are 4 steps away from finding a cure. The price of making one additional step towards a cure is 4 million pounds, and the cost of making two consecutive steps is 11 million pounds. At each stage a company chooses to take either one step or two steps. They take turns to choose, with Alice's company going first. Whenever a step towards a cure is made by any company, both companies acquire that knowledge. The first company to make the final step will patent the cure valued at 20 million pounds.

(a) Describe this game using a tree, carefully labelling all its components. ***(7 marks)***

(b) Which company will patent the cure and what will be its profit? Explain your answer, preferably by annotating the tree in (a). ***(5 marks)***

(c) Describe the game in strategic form, find all its pure-strategy Nash equilibria and indicate which of these is subgame perfect. ***(8 marks)***

(ii) A game of *Chomp!* starts with a rectangular array of pieces. Two players alternate taking turns. A turn consists of a player choosing a piece and removing that piece and all other pieces above or to the right of the chosen piece. The player who takes the last piece loses the game. Prove that the player who goes first has a strategy that guarantees her victory. ***(5 marks)***

4 (i) Consider a 2-player game given in strategic form as (S, T, u_1, u_2).

(a) Define the *minimax values* of both players. ***(2 marks)***

(b) Define the *cooperative payoff region* of the game. ***(2 marks)***

(ii) Consider the 2-person game G given in tabular form as follows

	A	B
I	0, 0	0, 2
II	4, 5	2, 0

(a) If the game were played a known finite number of times, what would happen in this game? ***(2 marks)***

(b) Show that the point $(3, 3)$ is in the cooperative payoff region of G by writing it as a convex combination of payoffs. (Hint: notice that $(3, 3)$ is in the convex hull of $(0, 2)$, $(2, 0)$ and $(4, 5)$.) ***(2 marks)***

(c) Consider now the game G^∞ which consists of playing G repeatedly, and where the payoffs of the infinite game are the average payoffs. Describe, without proof, a Nash equilibrium that results in average payoff of 3 for both players. ***(6 marks)***

(iii) Consider the following two games, where Alice is the row player and Bob is the column player.

Game I

	L	R
U	1, 2	3, 2
D	2, -1	1, -2

Game II

	L	R
U	9, 6	-5, -10
D	-10, 7	5, 14

(a) Assume that Alice and Bob play either Game I or Game II, neither knowing which game they are playing. Both know that Game I is played with probability 3/4. Model this as a Bayesian game, and find the pure-strategy Bayes-Nash equilibria of this game. ***(5 marks)***

(b) Assume now that Alice knows which game she is playing but Bob does not – Bob only knows that Game I is played with probability 3/4. Model this as a Bayesian game, and find the pure-strategy Bayes-Nash equilibria of this game. ***(6 marks)***

End of Question Paper