MAS348 – 2016-17 Solutions

Question 1

1(i)(a)

We find person's j best response to the other players playing $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n$ by finding the value of x_j which maximizes $u(x_1, \dots, x_n)$.

To do so we compute

$$\frac{\partial u_j}{\partial x_j} = \frac{100}{\left(1 + \sum_{i=1}^n x_i\right)^2} - 1$$

and solve $\frac{100}{\left(1+\sum_{i=1}^n x_i\right)^2} - 1 = 0$ for x_j giving $\left(1 + \sum_{i=1}^n x_i\right)^2 = 100$ hence $\sum_{i=1}^n x_i = 9$. Since these derivatives are decreasing functions when $x_j \geq 0$, the critical points are indeed maximal points.

Any pure Nash equilibrium is given by a strategy profile $(x_1^*, \dots, x_n^*)$ such that $\sum_{i=1}^n x_i^* = 9$ and a symmetrical one is given by $x_1^* = \cdots = x_n^* = 9/n$.

At the symmetric Nash equilibrium, each person's utility is $100(9/10) - 9/n = 90 = 90 - 9/n$.

1(i)(b) We now look for an $h \geq 0$ which maximizes the function $u(h) = u_j(h, \dots, h) = 100nh/(nh+1) - h$. We compute

$$\frac{du}{dh} = 100\frac{n(nh+1) - n^2h}{(nh+1)^2} - 1 = 100\frac{n}{(nh+1)^2} - 1$$

and

$$\frac{d^2u}{dh^2} = -2 \times 100n\frac{n}{(nh+1)^3} < 0 \quad (h \geq 0)$$

so to find the value of $h \geq 0$ that maximizes $u(h)$ we solve $100\frac{n}{(nh+1)^2} - 1 = 0$ giving a unique positive solution $h = (10\sqrt{n} - 1)/n$.

When the agreement is honoured, everyone's utility will be

$$10\frac{10\sqrt{n} - 1}{\sqrt{n}} - \frac{10\sqrt{n} - 1}{n}.$$

[Common mistake: In (a) first assume that all players must work same number of hours, and then look for the NE of this *different game*, in effecting answering part (b).]

1(ii)(a) II weakly dominates III and B strictly dominates C.

1(ii)(b) We eliminate strictly dominated strategy C, and after doing so II strictly dominates III so we eliminate III as well. The resulting game is

	I	II
A	2, 1	3, 4
B	4, 3	1, 2

1(ii)(c) Iterative elimination of dominated strategies does not affect Nash equilibria, so we may carry our our calculations in the 2 by 2 game in (b) instead of the original one. We mark Alice's and Bob's best responses as follows

	I	II
A	2, 1	$\underline{3}$, $\underline{4}$
B	$\underline{4}$, $\underline{3}$	1, 2

and discover the pure-strategy Nash equilibria (B,I) and (A,II).

We now look for a mixed strategy profile $((p, 1-p), (q, 1-q))$ which is a Nash equilibrium. The Principle of indifference gives us the equations $2q+3(1-q) = 4q+(1-q)$ and $p+3(1-p) = 4p+2(1-p)$. We solve these and obtain $p = 1/4$ and $q = 1/2$.

Question 2

(2)(i)(a) Write $S = \{s_1, \dots, s_m\}$ and $T = \{t_1, \dots, t_n\}$. Then Δ^R consists of all probability vectors $\{(p_1, \dots, p_m)^T \,|\, 0 \le p_1, \dots, p_m \le 1, p_1 + \cdots + p_m = 1\}$ and Δ^C consists of all probability vectors $\{(q_1, \dots, q_n)^T \,|\, 0 \le q_1, \dots, q_n \le 1, q_1 + \cdots + q_n = 1\}$.

(2)(i)(b) The value of the game V is the common value of

$\min_{q \in \Delta^C} \max_{p \in \Delta^R} u(p, q) = \max_{p \in \Delta^R} \min_{q \in \Delta^C} u(p, q)$. [Also, V is the value of a game if the row player has a strategy which guarantees her at least V and the column player has a strategy that guarantees him not to loose more than V]

(2)(i)(c) A strategy p^* for the row player is optimal if $\max_{p \in \Delta^R} \min_{q \in \Delta^C} u(p, q) = \min_{q \in \Delta^C} u(p^*, q)$, and a strategy q^* for the column player is optimal if $\min_{q \in \Delta^C} \max_{p \in \Delta^R} u(p, q) = \max_{p \in \Delta^R} u(p, q^*)$.

(2)(i)(d) Choose any $p \in \Delta^R$ and $q \in \Delta^C$.

$$u(p^*, q) \ge \min_{q \in \Delta^C} u(p^*, q) = \max_{p \in \Delta^R} \min_{q \in \Delta^C} u(p, q) = \min_{q \in \Delta^C} \max_{p \in \Delta^R} u(p, q) = \max_{p \in \Delta^R} u(p, q^*) \ge u(p, q^*)$$

and in particular $u(p^*, q) \ge u(p^*, q^*)$ and $u(p, q^*) \le u(p^*, q^*)$, hence (p^*, q^*) is a Nash equilibrium.

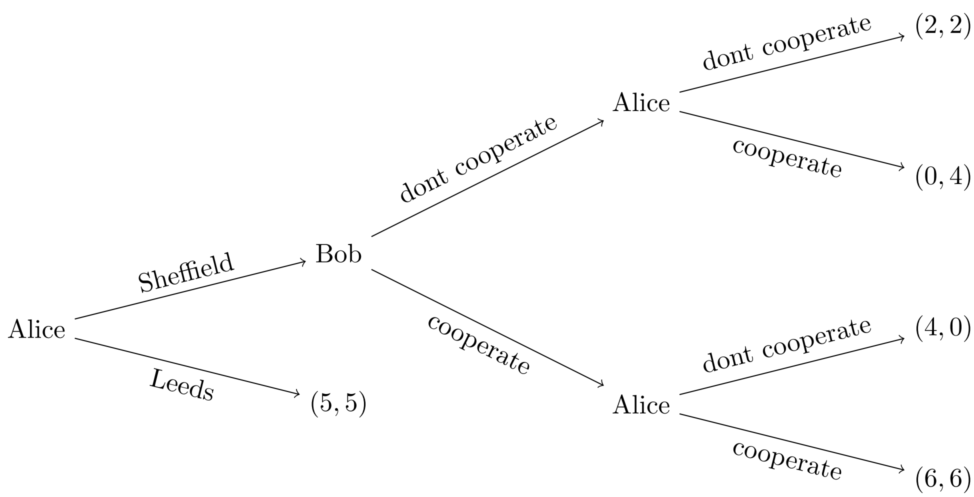
(2)(ii) Call the $n \times n$ magic square matrix M, and let c denote the sum of the elements in each of its rows and columns. Let $u \in \mathbb{R}^n$ be the column vector whose coordinates are all equal to 1. Since $u^T M = c u^T$, and letting $p = u/n$ we see that p is a probability vector and hence a mixed strategy for the row player, and $p^T M = (c/n) u^T$ so no matter what the column player plays, the row player's payoff will be c/n.

Similarly, $Mu = cu$, is also a mixed strategy for the column player, and $Mp = (c/n)u$ so no matter what the row player plays, the column player's payoff will be $-c/n$.

Now the row player has a strategy that guarantees her at least c/n and the column player has a strategy that guarantees him not to loose more than c/n. We deduce that the value of the game is c/n and that (p, p) is an optimal strategy profile.

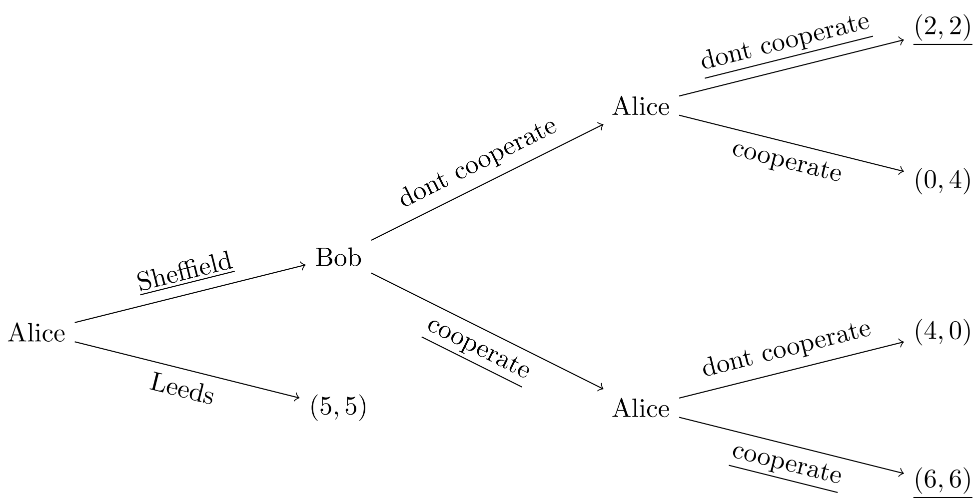
Question 3

(3)(i)(a)



where payoffs are in millions of pounds.

(3)(i)(b) A simple backward induction yields



Thus Alice will stay in Sheffield, Bob and Alice will cooperate and both will earn 6 million pounds.

(3)(i)(c) Alice has three decisions to make: Shefield or Leeds, cooperate or dont cooperate, when Bob cooperated, cooperate or dont cooperate, when Bob did not cooperate, and we list these as $[S,C,C],[S,C,D],[S,D,C],[S,D,D],[L,C,C],[L,C,D],[L,D,C],[L,D,D]$. Bob's strategies are *cooperate* and *dont cooperate*.

The strategic form of the game is given in tabular form as follows:

	cooperate	*dont cooperate*
$[S,C,C]$	(6,6)	(0,4)
$[S,C,D]$	(6,6)	(2,2)
$[S,D,C]$	(4,0)	(0,4)
$[S,D,D]$	(4,0)	(2,2)
$[L,C,C]$	(5,5)	(5,5)
$[L,C,D]$	(5,5)	(5,5)
$[L,D,C]$	(5,5)	(5,5)
$[L,D,D]$	(5,5)	(5,5)

	cooperate	*dont cooperate*
$[S,C,C]$	($\underline{6}$,$\underline{6}$)	(0,4)
$[S,C,D]$	($\underline{6}$,$\underline{6}$)	(2,2)
$[S,D,C]$	(4,0)	(0,$\underline{4}$)
$[S,D,D]$	(4,0)	(2,$\underline{2}$)
$[L,C,C]$	(5,$\underline{5}$)	($\underline{5}$,$\underline{5}$)
$[L,C,D]$	(5,$\underline{5}$)	($\underline{5}$,$\underline{5}$)
$[L,D,C]$	(5,$\underline{5}$)	($\underline{5}$,$\underline{5}$)
$[L,D,D]$	(5,$\underline{5}$)	($\underline{5}$,$\underline{5}$)

and we identify 6 Nash Equilibria, ([S,C,D], *cooperate*) is subgame-perfect while ([S,C,C], *cooperate*), ([L,C,C],*dont cooperate*), ([L,C,D], *dont cooperate*), ([L,D,C], *dont cooperate*), ([L,D,D], *dont cooperate*) are not.

(3)(ii) (a) The rank of the game is the length of the longest path in T.

(b)

We proceed by induction on the rank r of T. If $r = 1$, T consists of a root and leaves only and either player I can choose a leaf with outcome in S or she is forced to choose a leaf with outcome not in S.

Assume now that $r > 1$ and that the result holds for all games with trees of rank less than r. Let $G_1, \ldots, G_n$ be the subgames resulting after player I makes her move, and let $T_1, \ldots, T_n$ be the corresponding trees. Note that the ranks of $T_1, \ldots, T_n$ are less than r, and we may apply the induction hypothesis to each of $G_1, \ldots, G_n$. We have two cases: either

(i) Player II can force an outcome not in S in each one of the games $G_1, \ldots, G_n$, or

(ii) for some G_i, player II cannot force an outcome not in S.

If (i) holds, conclusion (b) follows, while if (ii) holds player I has a strategy which starts with a move to game G_i which forces an outcome in S.

Question 4

(4)(i)(a) The expected payoff for both players is $\sum_{i=0}^{\infty} 2(1/2)^i = 4$.

(4)(i)(b) Bob's best response to Alice's II is B, yielding a payoff of 1 in each repetition of the game. Thus Bob's expected payoff is $\sum_{i=0}^{\infty}(1/2)^i = 2$.

(4)(i)(c) This is not a Nash equilibrium because if Alice changes her strategy to playing II at every stage she would increase her expected payoff from 4 to $\sum_{i=0}^{\infty} x(1/2)^i = 2x > 4$.

(4)(i)(d) The strategy profile (II, B) is a Nash equilibrium of G, hence it is a Nash equilibrium for the repeated game.

(4)(i)(e) Consider Alice's strategy $\mathcal{A}$ consisting of playing I as long as Bob did not ever play B, and playing II forever once Bob played B. Consider Bob's strategy $\mathcal{B}$ consisting of playing A as long as Alice did not ever play II, and playing B forever once Alice played II. The strategy profile ($\mathcal{A}$, $\mathcal{B}$) results in an expected payoff of 4. If either player deviates at the nth repetition of this game, their expected payoff will be at most $\sum_{i=0}^{n-2} 2(1/2)^i + x(1/2)^{n-1} + \sum_{i=n}^{\infty}(1/2)^i$ and for the deviation not to pay off we must have $\sum_{i=0}^{n-2} 2(1/2)^i + x(1/2)^{n-1} + \sum_{i=n}^{\infty}(1/2)^i \leq \sum_{i=0}^{\infty} 2(1/2)^i$, i.e. $x(1/2)^{n-1} + \sum_{i=n}^{\infty}(1/2)^i \leq \sum_{i=n-1}^{\infty} 2(1/2)^i$, which yields $x + 1 \leq 4$ and hence $x \leq 3$.

(4)(ii)(a) There are two states in the world: Up and Down. Alice has two types A_U and A_D, Bob has one type. Alice has 9 strategies $\{N, R, E\} \times \{N, R, E\}$ where N stands for abstaining, R for voting remain, E for voting exit, and $[\alpha, \beta]$ denotes A_U choosing α and A_D choosing β. Bob has 3 strategies $\{N, R, E\}$.

$P(Up|A_U) = P(Down|A_D) = 1$, $P(Up|A_D) = P(Down|A_U) = 0$, $P(Up|Bob) = 9/10$, $P(Down|Bob) = 1/10$.

(4)(ii)(b)

We compile the expected payoffs of all relevant strategy profiles as follows

	R	E	N
[N, N]	[1 ,0],-		[1/2 , 1/2],-
[N, R]	[1 ,0],-		[1/2 , 0],-
[N, E]	[1 , 1/2], 19/20,	[-, -], 1/10	[1/2 , 1], 11/20
[R, N]	[1 , 0],-		[1 , 1/2],-
[R, R]	[1 , 0],-		[1, 0],-
[R, E]	[1 , 1/2], 19/20	[-, -], 11/20	[1 ,1], 1
[E, N]	[1/2 , 0],-		[0 , 1/2],-
[E, R]	[1/2 , 0],-		[0 , 0],-
[E, E]	[1/2 , 1/2],-		[0 , 1],-

and find best responses to Alice's [N, E] and [R, E] and to Bob's R and N

	R	E	N
[N, N]	[$\boxed{1}$,0],-		[1/2 , 1/2],-
[N, R]	[$\boxed{1}$,0],-		[1/2 , 0],-
[N, E]	[$\boxed{1}$, $\boxed{1/2}$], $\boxed{19/20}$,	[-, -], 1/10	[1/2, $\boxed{1}$], 11/20
[R, N]	[$\boxed{1}$, 0],-		[$\boxed{1}$, 1/2], -
[R, R]	[$\boxed{1}$, 0],-		[$\boxed{1}$, 0], -
[R, E]	[$\boxed{1}$, $\boxed{1/2}$], 19/20	[- ,-], 11/20	[$\boxed{1}$,$\boxed{1}$], $\boxed{1}$
[E, N]	[1/2 , 0],-		[0 , 1/2], -
[E, R]	[1/2 , 0],-		[0 , 0], -
[E, E]	[1/2 , $\boxed{1/2}$],-		[0 , $\boxed{1}$], -

uncovering both Bayes-Nash equilibria.