Cooperative games– pure strategies

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Games in strategic form

Games consist of players, available actions, and utilities of outcomes.

Definition: A *strategic form* of an n-person game consists of sets $S_1, S_2, \ldots, S_n$ (the sets of *actions* or *strategies*) and functions $u_i : S_1 \times S_2 \times \ldots S_n \to \mathbb{R}$ for all $1 \leq i \leq n$ (the utility functions). Elements in $S_1 \times S_2 \times \cdots \times S_n$ are called *strategy profiles*.

The strategic form of a Prisoners' Dilemma

Here $n = 2$, $S_1 = S_2 = \{\text{confess}, \text{do not confess}\}$, and $u_1 = u_2$ take values

$u_1(\{\text{confess}, \text{confess}\}) = u_2(\{\text{confess}, \text{confess}\}) = -5$,
$u_1(\{\text{do not confess}, \text{do not confess}\}) =$
$u_2(\{\text{do not confess}, \text{do not confess}\}) = -1$,
$u_1(\{\text{confess}, \text{do not confess}\}) = 10$,
$u_2(\{\text{confess}, \text{do not confess}\}) = -10$,
$u_1(\{\text{do not confess}, \text{confess}\}) = -10$,
$u_2(\{\text{do not confess}, \text{confess}\}) = 10$.

Dominant strategies

Definition Consider a game in strategic form $(S_1, S_2, \ldots, S_n, u_1, u_2, \ldots, u_n)$.
A strategy $s_i \in S_i$ *dominates* $s_i' \in S_i$ if

$$u_i(s_1, \ldots, s_i, \ldots, s_n) > u_i(s_1, \ldots, s_i', \ldots, s_n)$$

for all

$$s_1 \in S_1, \ldots, s_{i-1} \in S_{i-1}, s_{i+1} \in S_{i+1}, \ldots, s_n \in S_n.$$

We denote this $s_1 \gg s_1'$.
If $s_i \gg s_i'$, then it cannot possibly be advantageous ever for player to choose strategy s_i', because choosing s_i would always fare better, no matter what other players do!

Example: iterative elimination of dominated strategies

	l	r
U	0, 0	-1,-1
D	-3, 3	1, 1

$l \gg r$, and we can delete r!

	l
U	0, 0
D	-3, 3

Now $U \gg D$, and so Alice plays U.

I ask all students in this class to choose an integer between 1 and 100. The person whose number is closest to 2/3 of the average will receive £5. Which number do you choose?

Let x denote 2/3 of the average of the class choices.

$x < 67$, would anyone choose ≥ 68?

Now we know that $x < 45$, would anyone choose ≥ 45?

Now we know that $x \leq 30$, would anyone chose > 30?

We can continue this process until we are left with the single strategy of choosing 1.

(Why is this argument likely to fail in real life?)

Example: The Median Voter Theorem

Consider two candidates running for election, and assume that the electors care only about one issue which can be quantified with one number X (e.g., tax rates, immigration quotas, etc.)
The candidates, being politicians, will adopt the position which is most likely to get them elected. The Median Voter Theorem states that the winning strategy is to adopt the position of the median voter.
To see this assume that voters vote for the candidate whose position is closest to theirs (and if there is a tie they vote to either candidate with probability 50% , and assume, for simplicity, that the candidates can adopt the positions of the 10% centile, 20% centile, ... up to the 90% centile.

The game now looks as follows:

	10%	20%	30%	40%	50%	60%	70%	80%	90%
10%	50%	15%	20%	25%	30%	35%	40%	45%	50%
20%	85%	50%	25%	30%	35%	40%	45%	50%	55%
30%	80%	75%	50%	35%	40%	45%	50%	55%	60%
40%	75%	70%	65%	50%	45%	50%	55%	60%	65%
50%	70%	65%	60%	55%	50%	55%	60%	65%	70%
60%	65%	60%	55%	50%	45%	50%	65%	70%	75%
70%	60%	55%	50%	45%	40%	35%	50%	75%	80%
80%	55%	50%	45%	40%	35%	30%	25%	50%	85%
90%	50%	45%	40%	35%	30%	25%	20%	15%	50%

(entries in the table denote Alice's share of the vote.) Alice's 20% strategy dominates her 10% strategy, her 80% strategy dominates her 90% strategy. Eliminate these and similarly Bob's 10% and 90% strategies.

Now, (but not before!) the 30% strategies dominate the 20% strategies, and the 70% strategies dominate the 80% strategies–

	20%	30%	40%	50%	60%	70%	80%
20%	50%	25%	30%	35%	40%	45%	50%
30%	75%	50%	35%	40%	45%	50%	55%
40%	70%	65%	50%	45%	50%	55%	60%
50%	65%	60%	55%	50%	55%	60%	65%
60%	60%	55%	50%	45%	50%	65%	70%
70%	55%	50%	45%	40%	35%	50%	75%
80%	50%	45%	40%	35%	30%	25%	50%

Now, (but not before!) the 30% strategies dominate the 20% strategies, and the 70% strategies dominate the 80% strategies–eliminate the dominated strategies. We eventually end up with a single 50% strategy for both candidates.

Weakly dominant strategies

Definition

Consider a game in strategic form $(S_1, S_2, \dots, S_n, u_1, u_2, \dots, u_n)$.
A strategy $s_i \in S_i$ *weakly dominates* $s'_i \in S_i$ if

$$u_i(s_1, \dots, s_i, \dots, s_n) \geq u_i(s_1, \dots, s'_i, \dots, s_n)$$

for all

$$s_1 \in S_1, \dots, s_{i-1} \in S_{i-1}, s_{i+1} \in S_{i+1}, \dots, s_n \in S_n.$$

Example

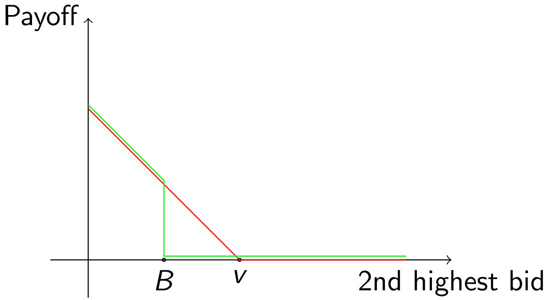
	easy	hard
easy	1, 1	1, 1
hard	0, 2	2, 0

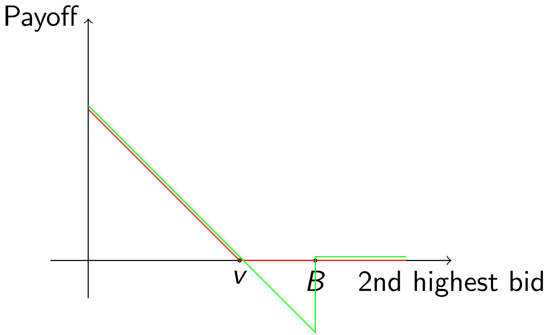
Example: Sealed-bid second-price auctions

An auction is a game played by potential buyers of an item whose outcome consists of a player and the price the player pays for the item.

For example sellers in *Ebay* sell their items in a process which is essentially equivalent to a *sealed-bid second-price auction*: in such an auction each bidder bids an amount of money *without knowing the other bids.* The item is then sold to the highest bidder at the *price bid by the second highest bidder.* If the value of the item to the winner of the auction is v and the second highest bid was x, the winner's payoff is $v - x$, and everyone else's payoff is zero.

Proposition. Bidding one's valuation of the item is a weakly dominant strategy.


Payoff
B
v
2nd highest bid


Payoff
v
B
2nd highest bid

Proof. Consider a bidder who values the item at v, and consider his strategies B which consist of bidding B. Let x be the second highest bid.
Bidding B produces the payoff

$$P_B(x) = \begin{cases} 0, & \text{if } x > B, \\ v - x & \text{if } x < B, \end{cases}$$

(note the tie breaker when $B = x$).
Now note that $P_v(x) \geq P_B(x)$ for all x:

- if $B \leq v$ then $P_B(x) = P_v(x) = v - x$ for $x \leq B$, and $P_B(x) = 0 \leq P_v(x)$ for $x > B$,
- if $B \geq v$ then $P_B(x) = P_v(x) = v - x$ for $x \leq v$, $P_B(x) = v - x \leq 0 = P_v(x)$ for $v \leq x \leq B$, and $P_B(x) = 0 = P_v(x)$ for $x > B$.

Pareto optimality

What is the correct notion of "solution" of a game which is not Dominant solvable? Maybe outcomes which make everyone as happy as possible?

Definition Consider a game in strategic form
$(S_1, \dots S_n, u_1, \dots, u_n)$.
A strategy profile is *Pareto optimal* if there is no other strategy profile that all players prefer,i.e., $(s_1, \dots s_n) \in S_1 \times \cdots \times S_n$ is *Pareto optimal* if for any other $(s'_1, \dots s'_n) \in S_1 \times \cdots \times S_n$ there is a $1 \leq j \leq n$ such that $u_j(s_1, \dots, s_n) > u_j(s'_1, \dots, s'_n)$.
Pareto optimal strategies, even if they exist, are not stable! For example, the solution of the Prisoners' Dilemma is not Pareto optimal. So Pareto optimality is not very useful for our purposes.

Stabiliy. Example: the investment game

We are all given the option of investing £10. If at least 90% of us choose to invest, we get back £20, otherwise we lose our £10 investment.
Do you invest?
Now lets do it for real– do you invest?
This poses a coordination problem: similar to adoption of an industry standard (VHS versus Betamax), joining a social network software (facebook versus myspace), bank runs!

Stabitity: best responses and Nash equilibria

Think of players who can change their strategies once their opponents strategy is revealed. A stable pair of strategies would be one that neither player would change their choice after learning the other player's move. We call such a pair of strategies a *no regrets* equilibrium.

Example

$$\begin{tikzcd}[ampersand replacement=\&]
 \& l \& r \\
U \& 3,3 \arrow[r] \arrow[d] \& -1,5 \arrow[d] \\
D \& 5,-1 \arrow[r] \& 0,0
\end{tikzcd}$$

The pair of strategies Dr is the only pair which neither player would want to move away from. Notice that both players would be better off if Ul were played.

Example: no stable strategy profile

$$\begin{tikzcd}[row sep=small]
 & l & r \\
U & 2,4 \arrow[d] & 1,0 \arrow[l] \\
D & 3,1 \arrow[r] & 0,4 \arrow[u]
\end{tikzcd}$$

Best response

Definition. Consider the game n-person game $(S_1, \ldots, S_n, u_1, \ldots, u_n)$.
An action $s_i \in S_i$ is a *best response* to a given set of actions $\{s_j \in S_j\}_{j \neq i}$ if
$u_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \geq u_i(s_1, \ldots, s_{i-1}, s, s_{i+1}, \ldots, s_n)$ for all $s \in S_i$.

Example.

	l	m	r
U	1, $\underline{3}$	$\underline{4}$, 2	2, 2
M	$\underline{4}$, 0	0, $\underline{3}$	4, 1
D	2, 5	3, 4	$\underline{5}$, $\underline{6}$

The best responses to strategies l, m and r are M, U and D, respectively. The best responses to strategies U, M and D are l, m and r, respectively.

Nash equilibrium

Definition. Consider the game n-person game $(S_1, \ldots, S_n, u_1, \ldots, u_n)$.
A strategy profile $(s_1, \ldots, s_n)$ is a *Nash equilibrium* if for all $1 \leq i \leq n$, s_i is a best response to $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n$.

Example.

	l	m	r
U	1, $\underline{3}$	$\underline{4}$, 2	2, 2
M	$\underline{4}$, 0	0, $\underline{3}$	4, 1
D	2, 5	3, 4	$\underline{5}$, $\underline{6}$

Here Dr is a Nash equilibrium.

Proposition. A Nash equilibrium is a no-regrets equilibrium.

Example: A coordination game with multiple Nash equilibria

	a	b
I	<u>0</u>, <u>0</u>	-1,-3
II	-3, -1	<u>1</u>, <u>1</u>

Both Ia and IIb are Nash equilibria!

Example: Best response functions

Alice and Bob collaborate on a project, each devoting up to 4 hours to it. If Alice works $0 \leq x \leq 4$ hours and Bob works $0 \leq y \leq 4$, their combined profit will be $4(x + y) + xy$ which they share equally. For both the cost of devoting h hours to the project is h^2, hence if they devote x and y hours, respectively, their total utilities are $A(x, y) = 2(x + y) + xy/2 - x^2$ and $B(x, y) = 2(x + y) + xy/2 - y^2$, respectively.

Note that this is our first game with infinite sets of actions!

$A(x,y) = 2(x+y) + xy/2 - x^2$, $B(x,y) = 2(x+y) + xy/2 - y^2$.
Find Alice's best response to Bob's y_0 hours' work, i.e., find the value of x which maximizes $A(x, y_0)$: $\partial A/\partial x = 2 + y/2 - 2x$ and $\partial^2 A/\partial x^2 = -2 < 0$, maximum is at $x = 1 + y_0/4$.
Similarly, for a fixed $0 \leq x_0 \leq 4$, $B(x_0, y)$ is maximized at $y = 1 + x_0/4$.
If we want x_0 and y_0 to be best responses to each other we solve

$$\begin{cases} x_0 = 1 + y_0/4 \\ y_0 = 1 + x_0/4 \end{cases}$$

and obtain $x_0 = y_0 = 4/3$.
Notice that this solution is not Pareto optimal: If both Alice and Bob devote h hours to the project, they both receive utility worth $h(4 - h/2)$ which is maximized when $h = 4$. They would be better off is someone could impose this better arrangement.

Elimination of dominated strategies does not loose Nash equilibria

Proposition. Consider a game $(S_1, \ldots, S_n, u_1, \ldots, u_n)$. If $s_i, s_i' \in S_i$ and $s_i \gg s_i'$, s_i' cannot occur in a Nash equilibrium.
Proof. Let $s_j \in S_j$ for all $j \neq i$. Since
$u_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) > u_i(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n)$
for $s_j \in S_j$ $(j \neq i)$, s_i' cannot be a best response to this $\{s_j \in S_j\}_{j \neq i}$ and so it cannot be a best response to anything.

Example: weak domination and Nash equilibria

Consider the game

	l	r
U	$\underline{1}$, $\underline{1}$	100, 0
D	0, 100	$\underline{100}$, $\underline{100}$

We see that U weakly dominates D and l weakly dominates r. We can also verify that both Ul and Dr are Nash equilibria. If we discard the weakly dominated strategies we loose the Pareto optimal Nash equilibrium!

Weird Example: Braess's Paradox

A large number of drivers travel from A to B either through C or D. Travel times are $AC = f, CB = 1.2, AD = 1.2, DB = f'$ where f, f' is the proportion of drivers going through C and D.
Can this be modelled as a game? What are its Nash equilibria?
There are many equilibria corresponding to is $f = 1/2$ giving a travel time of 1.7.
Now if a road from C to D taking 0.1 hours is opened, all will travel through it and resulting time will be 2.1!