Nash equilibria in Economics: monopolies, duopolies and oligopolies

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How do producers control their profits?

- Control of production levels (*Cournot model*).
- Control of prices (*Bertrand model*).

In practice both approaches are used by firms, but we study these in isolation.
Throughout this chapter we will consider a product whose cost of production per unit is c.

The Cournot model

Assumption: the price p of each unit of product is a (decreasing) function of the total supply q; specifically we will set $p(q) = a - bq$ in our discussion below.
We must have $p > c$ otherwise there is no profit to be made and hence no production, so $a > c$.

Monopolies

A monopolist faces a simple problem: maximize
$f(q) = (p(q) - c)q = (a - bq - c)q$.
The maximum occurs at $q = (a - c)/2b$, resulting in a price of
$p = a - (a - c)/2 = (a + c)/2$ profit of $(a - c)^2/4b$.

Duopolies

We model two firms producing an identical product.
These two players's strategies amount to deciding their production levels q_1 and q_2.
The price per unit of product is given by $p = a - b(q_1 + q_2)$.
The profits then are

$$(p - c)q_1 = (a - b(q_1 + q_2) - c)q_1 = aq_1 - bq_1^2 - bq_1q_2 - cq_1$$

and $(p - c)q_2 = aq_2 - bq_2^2 - bq_1q_2 - cq_2$.
We find Nash equilibria by finding the firms' best responses. If company 2 produces q_2 units, company 1 needs to maximize $aq_1 - bq_1^2 - bq_1q_2 - cq_1$:
differentiate with respect to q_1, set that to be zero and solve for q_1: from $a - 2bq_1 - bq_2 - c = 0$ and we obtain
$\mathrm{BR}_1(q_2) = (a - bq_2 - c)/2b = (a - c)/2b - q_2/2$.
Similarly, the second company's best response to company 1 producing q_1 units is $BR_2(q_1) = (a - c)/2b - q_1/2$.

To find the Nash equilibrium we solve the system of equations $q_1^* = BR_1(q_2^*), q_2^* = BR_2(q_1^*)$, and by symmetry we see that $q_1^* = q_2^*$.
We now obtain $q_1^* = q_2^* = (a - c)/3b$.
The price is now
$p = a - 2b(a - c)/3b = a - 2(a - c)/3 = (a + 2c)/3$.
Since $a > c$, the duopoly price is lower than the monopoly price of $(a + c)/2$; further, total production is greater than in a monopoly.
This is partly why societies regulate against monopolies: monopolies produce less and sell their products at higher prices to the detriment of consumers.

The total profits are now

$$(p-c)(q_1+q_2) = ((a+2c)/3-c)\,\frac{2(a-c)}{3b} = \frac{2}{9}\frac{(a-c)^2}{b}$$

and so the profit for each company is $(a-c)^2/9b$, less than half the monopoly profit.

Notice that if both companies collude to form a *cartel* and agree to produce $(a-c)/4b$ each, their total profits increase to the monopoly profit

$$(a-(a-c)/2-c)(a-c)/2b = (a-c)^2/4b.$$

Oligopolies and perfect competition

We now model n firms producing an identical product. As before, the firms decide their production levels $q_1, \ldots, q_n$.
The price per unit of product is now given by
$p = a - b(q_1 + \cdots + q_n)$ and the profits for the ith firm are

$$f_i(q_1, \ldots, q_n) = (p - c)q_i = (a - b(q_1 + \cdots + q_n) - c)q_i.$$

To maximize this we compute

$$\frac{\partial f_i}{\partial q_i} = a - b(q_1 + \cdots + q_n) - c - bq_i$$

and set it to zero.

We now obtain a system of equations

$$\left\{\begin{array}{cccccccccc}
2q_1 & + & q_2 & + & \dots & q_{n-1} & + & q_n & = & (a-c)/b \\
q_1 & + & 2q_2 & + & \dots & q_{n-1} & + & q_n & = & (a-c)/b \\
 & & \vdots & & \dots & & & & & \\
q_1 & + & q_2 & + & \dots & q_{n-1} & + & 2q_n & = & (a-c)/b
\end{array}\right.$$

which has a unique solution

$$q_1^* = \cdots = q_n^* = \frac{a-c}{(n+1)b}.$$

The total production is

$$\frac{n}{n+1}\frac{a-c}{b}$$

giving a price $p = a - \dfrac{n}{n+1}(a-c) = (a+nc)/(n+1)$,
and total profits

$$\left(\frac{a+nc}{n+1} - c\right)\frac{n}{n+1}\frac{a-c}{b} = \left(\frac{a-c}{n+1}\right)^2 \frac{n}{b}.$$

To model *perfect competition* we take $n \to \infty$ to obtain total production $(a-c)/b$, price c and total profit 0!

The Bertrand model

Control prices directly, affecting production levels indirectly through varying demand.
Specifically, we will assume that firms can produce any amount of their products, and that consumers will buy only from the cheapest producer.
We continue to assume that $p = a - bq$, but now we take the price p to be the independent variable and obtain an expression for the demand $q = (a - p)/b$.

We model a *Bertrand duopoly*: the two players first set their prices p_1 and p_2.
The profit functions for both company are

$$f_1(p_1,p_2) = \begin{cases} (p_1-c)\,(a-p_1)/b & \text{if } p_1 < p_2 \\ (p_1-c)\,(a-p_1)/2b & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2 \end{cases}$$

$$f_2(p_1,p_2) = \begin{cases} (p_2-c)\,(a-p_2)/b & \text{if } p_2 < p_1 \\ (p_2-c)\,(a-p_2)/2b & \text{if } p_1 = p_2 \\ 0 & \text{if } p_2 > p_1 \end{cases}$$

The only Nash equilibrium here is $p_1^* = p_2^* = c$ for a profit of zero!
To see this, we note that if $c < p_1^* \leq p_2^*$, firm 2 profits by changing its price to a bit less than p_1^*, and similarly if $c < p_2^* \leq p_1^*$, firm 1 profits by changing its price to a bit less than p_2^*.