

Cooperative games– mixed strategies

Moty Katzman

February 9, 2015

Example: matching pennies

Alice and Bob both turn a penny with either heads or tails. If they match, Alice takes them, otherwise Bob takes them.

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

Alice hires you as her Game Theory adviser; what do you advise? Game Theory cannot possibly advise Alice to play H or T; Bob's adviser is just as knowledgeable, and if, say, the correct move for Alice is H, Bob will be instructed to play T and win.

There is no correct action here, unless . . . we expand the notion of a strategy: Alice is advised to choose her move *randomly*, say, play H with probability α and T with probability $1 - \alpha$. Similarly Bob will choose H with probability β and T with probability $1 - \beta$.

Expected utility

Assumption: Given the choice between two uncertain outcomes, players will choose the one with highest expected utility. (The Von-Neumann, Morgenstein expected utility hypothesis.)

Note: With certain outcomes, utility encoded rankings. Now the actual values of the utility function matter!

Mixed strategies and mixed-strategy Nash equilibria

Consider a game $(S_1, \dots, S_n, u_1, \dots, u_n)$.

Definition. A *mixed strategy* for player i is a function $p_i : S_i \rightarrow [0, 1]$ such that $\sum_{s_i \in S_i} p(s_i) = 1$. (We interpret $p(s_i)$ as the probability that player i plays s_i .) Given mixed strategies p_1, \dots, p_n for players $1, \dots, n$ we define

$$u_i(p_1, \dots, p_n) = \sum_{(s_1, \dots, s_n) \in S_1 \times \dots \times S_n} p_1(s_1) \dots p_n(s_n) u_i(s_1, \dots, s_n)$$

(which is the expected payoff for player i when these mixed strategies are played.)

Definition. Mixed strategies p_1, \dots, p_n for players $1, \dots, n$ are a *mixed strategy Nash equilibrium* of the game if for all $1 \leq i \leq n$ and any mixed strategy p'_i for player i we have

$$u_i(p_1, \dots, p_n) \geq u_i(p_1, \dots, p_{i-1}, p'_i, p_{i+1} \dots p_n).$$

(Each p_i gives a response with highest expected value to the other mixed strategies.)

Example: matching pennies

Alice plays p , $p(H) = \alpha$, $p(T) = 1 - \alpha$;

Bob plays q , $q(H) = \beta$, $q(T) = 1 - \beta$. We have

$$u_1(p, q) = \alpha\beta + (1-\alpha)(1-\beta) - \alpha(1-\beta) - (1-\alpha)\beta = (4\beta - 2)\alpha + 1 - 2\beta$$

$$u_2(p, q) = -u_1(p, q).$$

To maximize u_1 for a given β , we take $\alpha = 1$ if $\beta > 1/2$, $\alpha = 0$ if $\beta < 1/2$ and any α when $\beta = 1/2$. Similarly, u_2 is maximized by $\beta = 0$ if $\alpha > 1/2$, $\beta = 1$ if $\alpha < 1/2$ and any β when $\alpha = 1/2$.

The only mixed strategy Nash equilibrium occurs with $\alpha = \beta = 1/2$.

Example: The battle of the sexes

Alice and Bob decide to go on a date together: Alice suggests going to the pub and Bob suggests watching a film. Before they can agree, the MAS348 lecture is over and they rush to different lectures. Realizing they don't have a way to contact each other before their date, they have to decide independently where to go. They are playing the following game:

	Pub	Film
Pub	3,1	0,0
Film	0,0	1,3

This game has two Nash equilibria in pure strategies, namely (Pub, Pub) and $(Film, Film)$. Are there any other Nash equilibria?

	Pub	Film
Pub	3,1	0,0
Film	0,0	1,3

Suppose Alice goes to the pub with probability p and Bob goes to the film with probability q . Alice's expected utility is $3p(1 - q) + (1 - p)q = p(3 - 4q) + q$ and Bob's expected utility is $p(1 - q) + 3(1 - p)q = q(3 - 4p) + p$. To find Alice's best responses we distinguish between three cases:

$q < 3/4$ and Alice's expected utility is maximized when $p = 1$,

$q > 3/4$ and Alice's expected utility is maximized when $p = 0$,

$q = 3/4$ now Alice's expected utility is $3/4$, regardless of p .

Similarly, to find Bob's best responses we distinguish between three cases:

$p < 3/4$ and Bob's expected utility is maximized when $q = 1$,

$p > 3/4$ and Bob's expected utility is maximized when $q = 0$,

$p = 3/4$ now Bob's expected utility is $3/4$, regardless of q .

The strategy profile corresponding to $p = q = 3/4$ is a Nash equilibrium!

Note: in a Nash equilibrium each player had an expected utility independent of the strategy adopted by the other player— *this is not a coincidence*.

Notation. For any action s of a player in a game, \hat{s} denotes the mixed strategy which plays s with probability 1.

Theorem. (The Indifference Principle) Consider a game (S_1, S_2, u_1, u_2) . Let (p, q) be a mixed strategy profile which is a Nash equilibrium. For any $s \in S_1$ with $p(s) > 0$ we have $u_1(\hat{s}, q) = u_1(p, q)$.

Proof. Let $\text{Supp}(p) = \{s \in S_1 \mid p(s) > 0\}$ (the “support” of p). We have $u_1(p, q) = \sum_{t \in \text{Supp}(p)} p(t)u_1(\hat{t}, q)$.

If the theorem does not hold, there must be an $s \in \text{Supp}(p)$ for which $u_1(\hat{s}, q) > u_1(p, q)$, contradicting the fact that p is a best response to q . \square

Example: Rock-Scissors-Paper

Consider the game

	Rock	Scissors	Paper
Rock	0, 0	1, -1	-1, 1
Scissors	-1, 1	0, 0	1, -1
Paper	1, -1	-1, 1	0, 0

There are no dominant strategies, nor are there Nash equilibria in pure strategies. To find mixed strategy profiles which are Nash equilibria, find a mixed strategy (p_1, p_2, p_3) for Alice which gives Bob equal utility for all his pure strategies: this happens in $p_2 - p_3 = -p_1 + p_3 = p_1 - p_2 = 0$, which together with $p_1 + p_2 + p_3 = 1$ yields the mixed strategy $(1/3, 1/3, 1/3)$. Similarly, Bob's strategy is $(1/3, 1/3, 1/3)$.

Example: Alice and Bob play tennis

Bob is at the net and Alice needs to decide to hit the ball to Bob's right or left; and Bob needs to decide to jump to left or right. The probabilities of Bob responding are as follows:

	left	right
left	50%	20%
right	10%	80%

and we assume Alice and Bob are playing the following game

	left	right
left	0.5, 0.5	0.8, 0.2
right	0.9, 0.1	0.2, 0.8

	left	right
left	0.5, 0.5	0.8, 0.2
right	0.9, 0.1	0.2, 0.8

There are no dominant strategies, nor are there Nash equilibria in pure strategies.

When is $((p, 1 - p), (q, 1 - q))$ a Nash equilibrium?

this happens when $0.5p + 0.1(1 - p) = 0.2p + 0.8(1 - p)$, which gives $p = 0.7$ and $0.5q + 0.8(1 - q) = 0.9q + 0.2(1 - q)$, which gives $q = 0.6$.

Example: The attrition game

Consider the game

	Stay	Leave
Stay	-2, -2	1, 0
Leave	0, 1	0, 0

There are two pure-strategy Nash equilibria: (Stay, Leave) and (Leave, Stay). There is also a mixed-strategy Nash equilibrium: $((1/3, 2/3), (1/3, 2/3))$.

Change this to

	Stay	Leave
Stay	-2, -2	$x, 0$
Leave	0, 1	0, 0

for $x > 1$. We preserve the pure-strategy Nash equilibria, but the mixed-strategy Nash equilibrium is now: $((1/3, 2/3), (x/(2+x), 2/(2+x)))$.

Change this to

	Stay	Leave
Stay	-2, -2	1, 0
Leave	y, 1	y, 0

for $0 < y < 1$. How does the mixed-strategy Nash equilibrium change as y increases?

Dominating mixed strategies

Even when no action strictly dominates another, a mixed strategy might.

Consider the game

	L	R
U	3, 1	0, 2
M	1, 1	1, 0
D	0, 0	3, 1

None of the actions is dominated, but playing U and D with probability $1/2$ each dominates M.

After eliminating M, Bob's R dominates L, and we obtain a solution (D, R) for this game.

Existence of Nash equilibria

Example. (A game with no Nash equilibrium)

Two players choose an integer. The player with the biggest integer gets 1, the other 0, and in case of a tie both get 0.

Example. (Another game with no Nash equilibrium)

$n > 1$ players choose a number in the open interval $(0, 1)$ and the mean μ of these numbers is computed. The players closest to $\mu/2$ win, the others lose.

Theorem. (Nash's Theorem)

Any game $(S_1, \dots, S_n, u_1, \dots, u_n)$ with finite strategy sets has at least one Nash equilibrium.

The proof of this theorem is quite intricate!