MAS348 Game Theory
Solutions #1

1. c strictly dominates a, b and d. y strictly dominates z, y weakly dominates x, b weakly dominates d.

2.

	x	y	z
a	1,$\underline{2}$	2,$\underline{2}$	5,1
b	4,1	3,$\underline{5}$	3,3
c	$\underline{5}$,2	$\underline{4}$,$\underline{4}$	$\underline{7}$,0
d	2,3	0,4	$\underline{7}$,$\underline{5}$

strategy	best response(s)
a	x, y
b	y
c	y
d	z
x	c
y	c
z	c, d

There are two strategy profiles which are Nash equilibria: (c, y) and (d, z).

3. Call Alice's position y.

(a) Assume first that $y < x_1$, and Bob needs to choose position x_1 or x_2. A voter with position z will vote for x_2 versus y whenever $z > (x_2+y)/2$ but since $(x_2+y)/2 > (x_1+y)/2$ that voter would also support x_1 over y. Hence, when $y < x_1$, Bob never loses by adopting x_1 rather than x_2. Consider now the case where $y > x_1$: in this case Bob's x_1 gets a proportion of $x_1 > 1/2$ of the vote and wins, and so there is no point in changing to position x_2. If $y = x_1$, the election yields a tie, and if Bob were to switch to position x_2, he would lose to Alice since $x_2 > x_1$. We see that Bob's x_1 does at least as well as his x_2 against any position y adopted by Alice.

(b) Assume first that $y > x_1$, and Bob needs to choose position x_1 or x_2. A voter with position z will vote for x_2 versus y whenever $z < (x_2+y)/2$ but since $(x_2+y)/2 < (x_1+y)/2$ that voter would also support x_1 over y. Hence, when $y > x_1$, Bob never loses by adopting x_1 rather than x_2. Consider now the case where $y < x_1$: in this case Bob's x_1 gets a proportion of $1 - x_1 > 1/2$ of the vote and wins, and so there is no point in changing to position x_2. If $y = x_1$, the election yields a tie, and if Bob were to switch to position x_2, he would lose to Alice since $x_2 < x_1$. We see that Bob's x_1 does at least as well as his x_2 against any position y adopted by Alice.

Now (c) follows from (a) and (b).

4. Consider $0 < a_1 < a_2 \leq 1$. Against Bob's $b = 100 - a_1$, Alice's a_1 and a_2 yield payoff of a_1 and 0, respectively, whereas against Bob's $b = 0$ they yield a_1 and a_2. So a_1 does not weakly dominate a_2, nor does a_2 weakly dominates a_1. On the other hand $a = 0$ is weakly dominated, but not strongly dominated, by any other of Alice strategies, and similarly for Bob's strategy $b = 0$. This also shows that there are no dominated strategies.

   We now show that (a, b) is a Nash equilibrium if and only if $a + b = 100$ or $a = b = 100$. To see that such (a, b) is a Nash equilibrium, notice that increasing a or b results in a payoff of 0, and hence its not to anyone's advantage, whereas decreasing one's bid does not increase one's payoffs.

   On the other hand if $a + b < 100$, Alice can increase her payoff if she bids $100 - b > a$ so this (a, b) is not a Nash equilibrium. If $a + b > 100$, and, say, $a < 100$, Bob can increase his payoff by bidding $100 - a > 0$.

   There is no mathematical argument to single out any strategy as better than the others. However, give human nature, bidding 50 sounds like a fair and reasonable thing to do which is likely to be reciprocated by the other player.

5. # The set of all Bob's strategies S_2 is the set of all functions $f : [0, 100) \to [0, 100]$. Each such function f encodes the strategy "if Alice bid a then bid $b = f(a)$".

   Bob's strategy $f_1(a) = 100 - a$ dominates all other strategies f_2 for which $f_2(a) \neq 100 - a$ for all a and weakly dominates all strategies f_3 for which $f_3(a) = 100 - a$ for some, but not all $0 \leq a < 100$.

6. For x, voting for A is never less advantageous than voting for B, so this is a weakly dominating strategy; it is not strongly dominating because when y and z vote B, x's vote makes no difference.

   For y and z, voting for B is never less advantageous than voting for A, so these are weakly dominating strategies; it is not strongly dominating because the two other players vote A, the third vote makes no difference.

   The following strategy profiles are Nash equilibria: (A,A,A), (A,B,B), (B,B,B).

7. The expected profits of Alice and Bob are given by $A(a, b) = 10^6 a/(a+b) - a$ and $B(a, b) = 10^6 b/(a + b) - b$. To find the best response of Alice to Bob's b we solve $0 = \partial A/\partial a = 10^6 b/(a+b)^2 - 1$ giving us $\mathrm{BR}_A(b) = 1000\sqrt{b} - b$ and, similarly, Bob's best response to Alice's a is $\mathrm{BR}_B(a) = 1000\sqrt{a} - a$. We find the Nash equilibrium by solving the system of equations $a^* = \mathrm{BR}_A(b^*)$ and $b^* = \mathrm{BR}_B(a^*)$, i.e., $a^* = 1000\sqrt{1000\sqrt{(}a^*) - a^*} - (1000\sqrt{(}a^*) - a^*)$ and we get $a^* = 250,000$, and by symmetry we also get $b^* = 250,000$. However, an easier way to find the Nash equilibrium is to notice that the symmetry between Alice and Bob implies that the Nash equilibrium will occur at the same value for both, thus $a^* = b^*$ and we need to solve $a^* = 1000\sqrt{a^*} - a^*$ giving $a^* = 250,000$.

   So at Nash equilibrium Alice and Bob would have equal chances of getting the contract and their expected profit would be £250,000.

   If Alice and Bob spend almost nothing and share the profit they would get about £500,000 each.

8. Call the number of units manufactured by each manufacturer $x_1, \dots, x_n$. The profit function for the ith manufacturer is $p_i(x_1, \dots, x_n) = x_i e^{-(x_1+\cdots+x_n)}$. We calculate

$$\frac{\partial p_i}{\partial x_i} = e^{-(x_1+\cdots+x_n)}(1 - x_i), \quad \frac{\partial^2 p_i}{\partial x_i^2} = -e^{-(x_1+\cdots+x_n)}(2 - x_i)$$

and hence, no matter what the values of $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ are, p_i attains a maximum at $x_i = 1$.

If everyone follows this strategy, each will receive a profit of e^{-n}, whereas if they all produce $1/n$ units each, the profit would be $e^{-1}/n = 1/ne > e^{-n}$.

Finally, the function $f(x) = xe^x$ which equals the combined profits, has a maximum at $x = 1$.

9. One Nash equilibrium occurs when everyone chooses twice the average. In this case consider the best response r_i of the ith person given that the other chose

$$r_1, \dots, r_{i-1}, r_{i+1}, \dots r_k.$$

Write $S = r_1 + \cdots + r_k$; we have $2 \times S/k = r_i$ and if $r_1, \dots, r_k$ is a Nash equilibrium, then $r_i = 2 \times S/k$ for all $1 \leq i \leq k$ and $S = r_1 + \cdots + r_k = 2 \times S$ which implies that $S = 0$ and hence $r_1 = \cdots = r_k = 0$. A non-technical explanation of this is the following: at equilibrium, people would all chose twice the average, but it is impossible for everyone to choose twice the average, unless the average is zero, because then average would be twice the average!

Another Nash equilibrium could occur when someone chooses less than twice the average, but in that case that person must be choosing the largest number possible, i.e., 100, otherwise increasing the choice to the winning choice would be advantageous. Thus in this case the Nash equilibrium occurs when everyone chooses 100.

10. Now the profit functions are

$$(p - c_1)q_1 = (a - b(q_1 + q_2) - c_1)q_1 = aq_1 - bq_1^2 - bq_1q_2 - c_1q_1$$

and $(p - c_2)q_2 = aq_2 - bq_2^2 - bq_1q_2 - c_2q_2$.

We aim to find Nash equilibria, and so we first find the firms' best responses. If company 2 produces q_2 units, we need to maximize $aq_1 - bq_1^2 - bq_1q_2 - c_1q_1$: differentiate with respect to q_1, set that to be zero and solve for q_1: $a - 2bq_1 - bq_2 - c_1 = 0$ and we obtain $\mathrm{BR}_1(q_2) = (a - bq_2 - c_1)/2b = (a - c_1)/2b - q_2/2$. Similarly, the second company's best response to company 1 producing q_1 units is $BR_2(q_1) = (a - c_2)/2b - q_1/2$. To find the Nash equilibrium we solve the system of equations $q_1^* = BR_1(q_2^*), q_2^* = BR_2(q_1^*)$, i.e., $q_1^* = (a - c_1)/2b - q_2^*/2$, $q_2^* = (a - c_2)/2b - q_1^*/2$, We now obtain $q_1^* = (a - 2c_1 + c_2)/3b$ and $q_2^* = (a - 2c_2 + c_1)/3b$.

The total production is $q_1^* + q_2^* = (a - 2c_1 + c_2 + a - 2c_2 + c_1)/3b = (2a - c_1 - c_2)/3b$, the corresponding price is $a - b(2a - c_1 - c_2)/3b = (a + c_1 + c_2)/3b$ corresponding profits are

$$(p-c_1)q_1 = ((a+c_1+c_2)/3-c_1)(a-2c_1+c_2)/3b = (a-2c_1+c_2)(a-2c_1+c_2)/9b = (a-2c_1+c_2)^2/9b$$

and

$$(p - c_2)q_2 = (a - 2c_2 + c_1)^2/9b.$$

The total profits are now

$$(a - 2c_1 + c_2)^2/9b + (a - 2c_2 + c_1)^2/9b.$$

11. Now the profit functions are $A(q1, q2) = (a - b(2q_1 + q_2) - c)q_1$ and $B(q1, q2) = (a - b(q_1 + 2q_2) - c)q_2$. We obtain best response functions $BR_1(q_2) = -q_2/4 + (a - c_1)/4b$ and $BR_2(q_1) = -q_1/4 + (a - c_2)/4b$.

    We solve the system of equations

    $$\begin{cases} q_1^* = -q_2^*/4 + (a - c_1)/4b \\ q_2^* = -q_1^*/4 + (a - c_2)/4b \end{cases}$$

    which gives a Nash equilibrium $q_1^* = (3a + c_2 - 4c_1)/15b$, $q_2^* = (3a + c_1 - 4c_2)/15b$.

12. # Suppose that the firms collude to produce (q_1^*, q_2^*) and let C_1 and C_2 be the isoprofit curves of firms 1 and 2 passing through the point (q_1^*, q_2^*). If the curves are not tangent, i.e., if C_1 goes across C_2 as it passes through (q_1^*, q_2^*) then there are points on C_1 near (q_1^*, q_2^*) where the profits of firm 2 are higher; pick any of these points (q_1', q_2'). Changing production to (q_1', q_2') does not affect firm 1 (we moved along its isoprofit curve, but firm 2 would benefit from moving to this new level of production. So when C_1 and C_2 are not tangent at (q_1^*, q_2^*) , this production profile is not Pareto efficient.)