MAS348 Game Theory
Solutions #2

1. There are no pure-strategy Nash equilibria, so we look for mixed-strategy Nash equilibria of the form $((p, 1-p), (q, 1-q))$.

   If the row player mixes her strategies, then she is indifferent between her two possible actions and so $q + 2(1-q) = 2q + 1 - q$ and hence $q = 1/2$.

   If the column player mixes his strategies, then he is indifferent between his two possible actions and so $2p + 3(1-p)$=$p + 5(1-p)$ and hence $p = 2/3$.

2. The first game has payoff table

	l	r
L	0,1	1,0
R	3/4,1/4	0,1

   and it is easy to see that there is no pure-strategy Nash equilibrium. We look for a mixed Nash equilibrium $(p, 1-p)$ for Alice and $(q, 1-q)$ for Bob. Bob's pure strategies must be indifferent to Alice's NE strategy , i.e., $p + 1/4 \times (1-p) = 1-p$ and Alice's pure strategies must be indifferent to Bobs's NE strategy , i.e., $1 - q = 3/4 \times q$. We obtain $p = 3/7$ and $q = 4/7$. Notice that Alice kicks more often to the Right, even though she kicks better to the Left.

   The second game has payoff table

	l	r
L	0.8,0.2	1,0
R	$\alpha, 1-\alpha$	0.8, 0.2

   If $\alpha < 0.8$, Ll is a pure-strategy Nash equilibrium. If $\alpha = 0.8$ both Ll and Rl are pure-strategy Nash equilibria. With p and q as above, we look for mixed NE and obtain the conditions $0.2p + (1-\alpha)(1-p) = 0.2(1-p)$ and $0.8q + (1-q) = \alpha q + 0.8(1-q)$ giving $p = (5\alpha - 4)/(5\alpha - 3)$ and $q = 1/(5\alpha - 3)$. Now q is a probability when $\alpha \geq 0.6$ and for these α p is a probability when $\alpha \geq 0.8$. We conclude that for $0.8 \leq \alpha \leq 1$ we have a mixed NE

   $$\left(\frac{5\alpha-4}{5\alpha-3}, \frac{1}{5\alpha-3}\right), \left(\frac{1}{5\alpha-3}, \frac{5\alpha-4}{5\alpha-3}\right).$$

3. Denote the game (S_1, S_2, u_1, u_2). Let (p, q) be a mixed-strategy Nash-equilibrium, and suppose that pure strategies s and t are in the support of p and that s dominates t. Since s dominates t, $u_1(s, z) > u_1(t, z)$ for all $z \in S_2$ and since for all $z \in S_2$ $q(z) \geq 0$ and some of these values are positive, $\sum_{z \in S_2} q(z)u_1(s, z) > \sum_{z \in S_2} q(z)u_1(t, z)$.

   On the other hand, if s, t are in the support of p, the Indifference Principle implies $\sum_{z \in S_2} q(z)u_1(s, z) = \sum_{z \in S_2} q(z)u_1(t, z)$, a contradiction.

4. # Let $e_1, \dots, e_n$ denote the standard column vectors. Notice that the Indifference Principle implies that the values of $p^T U_1 e_i$ are the same for all $1 \le i \le n$ and that the values of $e_j^T U_1 q$ are the same for all $1 \le j \le n$, i.e., there exist constants c_1, c_2 such that $p^T U_1 = c_1 e^T$ and $U_2 q = c_2 e$ where $e = e_1 + \cdots + e_n$. We can now write $p^T = c_1 e^T U_1^{-1}$ and $q = c_2 U_2^{-1} e$. To completely determine p and q add the condition that these are probability vectors: compute $p'^T = e^T U_1^{-1}$ and $q' = U_2^{-1} e^t$; now take $p = p'/(p'_1 + \cdots + p'_n)$ and $q = q'/(q'_1 + \cdots + q'_n)$. These are the only possible Nash equilibrium with all strategies in their support (and give a Nash equilibrium if and only if these p and q are probability vectors).

5. # This game is described by a table with rows and columns indexed by $0, \dots, n-1$. The payoffs in entry (i,j) are

$$\begin{cases} (-i, N-j) & \text{if } i < j \\ (-i, -i) & \text{if } i = j \\ (N-i, -j) & \text{if } i > j \end{cases}$$

The fact that the row player mixes her strategies implies that each of her pure strategies does equally well against the column player's mixed strategy, i.e., the values of

$$(N-i) \times p_0 + (N-i) \times p_1 + \cdots + (N-i) \times p_{i-1} - i \times p_i - i \times p_{i+1} - \cdots - i \times p_{N-1}$$

are the same for all $0 \le i \le n-1$.

If we write this system for $N = 5$ we get the system of equations

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & -1 & -1 & -1 & -1 \\ 3 & 3 & -2 & -2 & -2 \\ 2 & 2 & 2 & -3 & -3 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_4 \end{bmatrix} = 0$$

which has a solution $p_0 = p_1 = \cdots = p_4 = 1$.

To see that this is a solution for general N we verify that

$$(N-i) \times 1 + (N-i) \times 1 + \cdots + (N-i) \times 1 - i \times 1 - i \times 1 - \cdots - i \times 1 = (N-i)i - i(N-i) = 0.$$

We make this solution into probabilities by dividing by n and we obtain a mixed NE $(1/n, \dots, 1/n), (1/n, \dots, 1/n)$.

6. We first show that there is no pure-strategy *symmetric* Nash equilibrium: if no one calls for help, then one of them can do so and receive a strictly higher payoff of $v - c > 0$ and if all call, then any one can deviate by not calling and receive a strictly higher payoff $v > v - c$. (Note that the situation in which precisely one person calls for help is an *asymmetric* Nash equilibrium).

   Thus, a symmetric equilibrium, if one exists, should be in mixed strategies. Let p be the probability that a person does not call for help. Consider bystander is payoff of this mixed strategy prole: when she doesn't call for help, her expected payoff is $p^{n-1} \times 0 + (1 - p^{n-1}) \times v = (1 - p^{n-1})v$, and when she does call for help, her payoff is $v - c$. The fact that bystander i mixes her strategies implies that she is indifferent between these two, hence $(1 - p^{n-1})v = v - c$, and $p = (c/v)^{1/(n-1)}$. The probability that no one calls for help is $p^n = (c/v)^{n/(n-1)}$.

7. The only saddle points (and pure strategy Nash equilibria) occur at $(1, 2)$ and $(3, 2)$.

8. There is no saddle point. To answer (b) we compute

$$[1/5, 2/5, 2/5] \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = [1/5, 2/5, 2/5] \begin{bmatrix} 1/3 \\ 0 \\ 1/3 \end{bmatrix} = 3/15$$

   Since

$$[1/5, 2/5, 2/5] \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} = [3/5, -3/5, 3/5]$$

   the column player plays his second strategy.

9. The gain for the row player from each of his pure strategies is $-3/5$, so any (pure or mixed) strategy fares equally well.

10. Notice that $u(s_1, t_1) \le u(s_1, t)$ for all $t \in T$ and in particular $u(s_1, t_1) \le u(s_1, t_2)$. Similarly $u(s, t_2) \le u(s_2, t_2)$ for all $s \in S$ and in particular $u(s_1, t_2) \le u(s_2, t_2)$. Similarly $u(s_2, t_2) \le u(s_2, t)$ for all $t \in T$ and in particular $u(s_2, t_2) \le u(s_2, t_1)$. Finally $u(s, t_1) \le u(s_1, t_1)$ for all $s \in S$ and in particular $u(s_2, t_1) \le u(s_1, t_1)$. Putting everything together we get $u(s_1, t_1) \le u(s_1, t_2) \le u(s_2, t_2) \le u(s_2, t_1) \le u(s_1, t_1)$ thus all these are equal.

    Now $u(s, t_2) \le u(s_2, t_2) = u(s_1, t_2)$ for all $s \in S$ and $u(s_1, t) \ge u(s_1, t_1) = u(s_1, t_2)$ for all $t \in T$.

11. The mixed strategy $(1/2, 1/2, 0)$ and the pure strategy C guarantee the row and column players, respectively, an expected payoff of 0.

    We deduce that the value v of the game satisfies $0 \ge v \ge 0$ and hence $v = 0$. The row player's mixed strategy (p_1, p_2, p_3) yields $-\delta p_3$ against against C, and so it cannot be optimal unless $p_3 = 0$. If $p_3 = 0$, the strategy $(1/2, 1/2, 0)$ is the only one which yields at least 0 against A and B.

    To find the column player optimal strategies we demand that they yield at most 0 against I, II and III, i.e., $q_1 - q_2 \le 0$, $-q_1 + q_2 \le 0$ and $\alpha(q_1 + q_2) - \delta q_3 \le 0$. This, together with $q_1 + q_2 + q_3 = 1$ gives $q_1 = q_2 \le \delta/2(\alpha + \delta)$. Thus the row player has a unique (mixed) optimal strategy whereas the set of optimal strategies for the column player has one pure strategy and infinitely many mixed strategies.

12. There are no dominated strategies, nor is there an optimal pure strategy profile.

Using our methods from cooperative games, we look for a row mixed strategy $(x, 0, 1-x)$ which makes Bob indifferent between A and B, i.e., $x = -x + 2(1-x)$, and we obtain $p^* = (1/2, 0, 1/2)$. We now look for a column mixed strategy $(y, 1-y, 0)$ which makes Alice indifferent between I and III, i.e., $y - (1-y) = 2(1-y)$ and we obtain $q^* = (3/4, 1/4, 0)$.

To verify that the strategy profile is optimal we compute

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3/4 \\ 1/4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ \mathbf{-3/2} \\ 1/2 \end{bmatrix}$$

and verify that $p^*(II) = 0$. We now compute

$$\begin{bmatrix} 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ -2 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \end{bmatrix}$$

and since $\max\{1/2, -3/2, 1/2\} = \min\{1/2, 1/2, 1/2\}$, this strategy profile is optimal.

13. Note that $\sigma^T A \sigma = -A$. If p is an optimal strategy for both players,

$$(\sigma p)^T A \sigma p = p^T \sigma^T A \sigma p = -p^T A p = 0$$

since the value of the game is zero. Thus $p' = \sigma p = (p_n, p_{n-1}, \ldots, p_1)^T$ is also optimal for both players and we take $q = (p + p')/2$: note that $p'^T A p = (p'^T A p)^T = p A^T p' = -pAp'$ and

$$q^T A q = \frac{1}{4}\left(p^T A p + p'^T A p' + p^T A p' + p'^T A p\right) = 0.$$

14. (a) The 100×100 matrix A of this game looks like

$$\begin{bmatrix} 0 & -1 & 2 & 2 & 2 & \ldots \\ 1 & 0 & -1 & 2 & 2 & \ldots \\ -2 & 1 & 0 & -1 & 2 & \ldots \\ -2 & -2 & 1 & 0 & -1 & \ldots \\ -2 & -2 & -2 & 1 & 0 & \ldots \\ & & & & & \ddots \end{bmatrix}$$

i.e., its (i,j) entry is 0 if $i = j$, 1 if $i = j+1$, -2 if $i > j+1$, -1 if $j = i+1$, and 2 if $j > i+1$. This game is symmetric because the players have identical strategy sets and the matrix A describing this game satisfies $A^T = -A$.

(b) Row 1 dominates all rows from 4 to 100, and column 1 dominates all columns from 4 to 100. After eliminating these we are left with the 3×3 matrix

$$B = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}$$

(c) Exercise 13 shows that both payers have same optimal strategy of the form $p = (t, 1-2t, t)^T$ and to find it and that the game has value 0. So we need to solve $Bp = 0$ which gives $p = (1/4, 1/2, 1/4)$.

15. # (a) The matrix A for this zero-sum game is lower triangular with 1 on the diagonal, -1 below the diagonal and 0 above it.

(b) Let (p, q) be an optimal strategy profile and assume that all pure strategies occur with positive probability. The Indifference Principle implies

$$\begin{cases} q_1 & = V \\ -q_1 + q_2 & = V \\ -q_1 - q_2 + q_3 & = V \\ & \vdots \\ -q_1 - q_2 - q_3 - \dots q_{n-1} + q_n & = V \end{cases} \quad \begin{cases} p_1 - p_2 - \dots - p_n & = V \\ p_2 - p_3 - \dots - p_n & = V \\ p_3 - p_4 - \dots - p_n & = V \\ & \vdots \\ p_n & = V \end{cases}$$

where V is the value of the game. The solutions of these are $V = 2/n(n+1)$, $q = (V, 2V, \dots, nV)$ and $p = (nV, (n-1)V, \dots, V)$. We note that $p \in \Delta^R$ and $q \in \Delta^C$.

(c) Let u be the payoff function of this game. Since we used the Indifference Principle to find p and q, we have $u(p, \widehat{1}) = u(p, \widehat{2}) = u(p, \widehat{3}) = V$ and $u(\widehat{1}, q) = u(\widehat{2}, q) = u(\widehat{3}, q) = V$. Now the value of the game is at least $\min\{u(p, \widehat{1}), u(p, \widehat{2}), \dots, u(p, \widehat{n})\} = V$ and at most $\max\{u(\widehat{1}, q), u(\widehat{2}, q), \dots, u(\widehat{n}, q)\} = V$ so the value of the game is indeed V and the strategy profile in (b) is indeed optimal.