MAS348 Game Theory
Solutions #2

1. There are no pure-strategy Nash equilibria, so we look for mixed-strategy Nash equilibria of the form $((p, 1-p), (q, 1-q))$.

   If the row player mixes her strategies, then she is indifferent between her two possible actions and so $q + 2(1-q) = 2q + 1 - q$ and hence $q = 1/2$.

   If the column player mixes his strategies, then he is indifferent between his two possible actions and so $2p + 3(1-p) = p + 5(1-p)$ and hence $p = 2/3$.

2. The tabular form of the game is

	Swerve	Don't Swerve
Swerve	0, 0	-1, 1
Don't Swerve	1, -1	-10, -10

   There are two pure-strategy Nash equilibria: (Don't Swerve, Swerve) and (Swerve, Don't Swerve). We look for mixed Nash-Equilibria $(p, 1-p)$ and $(q, 1-q)$ for the row and column players, respectively. Under the assumption that $0 < p, q < 1$ the indifference principle implies that $p \times 0 + (1-p) \times (-1) = p \times 1 + (1-p) \times (-10)$ and $q \times 0 + (1-q) \times (-1) = q \times 1 + (1-q) \times (-10)$. We solve and obtain $p = q = 9/10$.

3. The first game has payoff table

	l	r
L	0,1	1,0
R	3/4,1/4	0,1

   and it is easy to see that there is no pure-strategy Nash equilibrium. We look for a mixed Nash equilibrium $(p, 1-p)$ for Alice and $(q, 1-q)$ for Bob. Bob's pure strategies must be indifferent to Alice's NE strategy , i.e., $p + 1/4 \times (1-p) = 1 - p$ and Alice's pure strategies must be indifferent to Bobs's NE strategy , i.e., $1 - q = 3/4 \times q$. We obtain $p = 3/7$ and $q = 4/7$. Notice that Alice kicks more often to the Right, even though she kicks better to the Left.

   The second game has payoff table

	l	r
L	0.8,0.2	1,0
R	$\alpha, 1-\alpha$	0.8, 0.2

   If $\alpha < 0.8$, Ll is a pure-strategy Nash equilibrium. If $\alpha = 0.8$ both Ll and Rl are pure-strategy Nash equilibria. With p and q as above, we look for mixed NE and obtain the conditions $0.2p + (1-\alpha)(1-p) = 0.2(1-p)$ and $0.8q + (1-q) = \alpha q + 0.8(1-q)$ giving $p = (5\alpha - 4)/(5\alpha - 3)$ and $q = 1/(5\alpha - 3)$. Now q is a probability when $\alpha \geq 0.6$ and for these α p is a probability when $\alpha \geq 0.8$. We conclude that for $0.8 \leq \alpha \leq 1$ we have a mixed NE

$$\left(\frac{5\alpha-4}{5\alpha-3}, \frac{1}{5\alpha-3}\right), \left(\frac{1}{5\alpha-3}, \frac{5\alpha-4}{5\alpha-3}\right).$$

4. Denote the game (S_1, S_2, u_1, u_2). Let (p, q) be a mixed-strategy Nash-equilibrium, and suppose that pure strategies s and t are in the support of p and that s dominates t. Since s dominates t, $u_1(s, z) > u_1(t, z)$ for all $z \in S_2$ and since for all $z \in S_2$ $q(z) \geq 0$ and some of these values are positive, $\sum_{z \in S_2} q(z) u_1(s, z) > \sum_{z \in S_2} q(z) u_1(t, z)$.

   On the other hand, if s, t are in the support of p, the Indifference Principle implies $\sum_{z \in S_2} q(z) u_1(s, z) = \sum_{z \in S_2} q(z) u_1(t, z)$, a contradiction.

5. Note that first player's p is a best response the second player's q in G, if and only if it is a best response to q in $\overline{G}$ (because the first player has identical utility functions in both games). Thus we need to show that the second player's q is a best response to p in G, if and only if it is a best response to p in $\overline{G}$: for all second player's mixed strategies q',

$$\begin{aligned}
\sum_{s \in S, t \in T} p(s) q(t) w(s,t) \geq \sum_{s \in S, t \in T} p(s) q'(t) w(s,t) \quad &\iff \\
\sum_{s \in S, t \in T} p(s) q(t) (a v(s,t) + b) \geq \sum_{s \in S, t \in T} p(s) q'(t) (a v(s,t) + b) \quad &\iff \\
a \left(\sum_{s \in S, t \in T} p(s) q(t) v(s,t) \right) + b \geq a \left(\sum_{s \in S, t \in T} p(s) q'(t) v(s,t) \right) + b \quad &\iff \\
\sum_{s \in S, t \in T} p(s) q(t) v(s,t) \geq \sum_{s \in S, t \in T} p(s) q'(t) v(s,t) &
\end{aligned}$$

   where the last equivalence follows from the fact that $a > 0$.

6. # This game is described by a table with rows and columns indexed by $0, \ldots, n-1$. The payoffs in entry (i, j) are

$$\begin{cases} (-i, N-j) & \text{if } i < j \\ (-i, -i) & \text{if } i = j \\ (N-i, -j) & \text{if } i > j \end{cases}$$

   The fact that the row player mixes her strategies implies that each of her pure strategies does equally well against the column player's mixed strategy, i.e., the values of

$$(N-i) \times p_0 + (N-i) \times p_1 + \cdots + (N-i) \times p_{i-1} - i \times p_i - i \times p_{i+1} - \cdots - i \times p_{N-1}$$

   are the same for all $0 \leq i \leq n-1$.

If we write this system for $N = 5$ we get the system of equations

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & -1 & -1 & -1 & -1 \\ 3 & 3 & -2 & -2 & -2 \\ 2 & 2 & 2 & -3 & -3 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_4 \end{bmatrix} = 0$$

which has a solution $p_0 = p_1 = \cdots = p_4 = 1$.

To see that this is a solution for general N we verify that

$$(N-i)\times 1+(N-i)\times 1+\cdots+(N-i)\times 1-i\times 1-i\times 1-\cdots-i\times 1 = (N-i)i-i(N-i) = 0.$$

We make this solution into probabilities by dividing by n and we obtain a mixed NE $(1/n, \ldots, 1/n), (1/n, \ldots, 1/n)$.

7. We first show that there is no pure-strategy <u>symmetric</u> Nash equilibrium: if no one calls for help, then one of them can do so and receive a strictly higher payoff of $v - c > 0$ and if all call, then any one can deviate by not calling and receive a strictly higher payoff $v > v - c$. (Note that the situation in which precisely one person calls for help is an <u>asymmetric</u> Nash equilibrium).

   Thus, a symmetric equilibrium, if one exists, should be in mixed strategies. Let p be the probability that a person does not call for help. Consider bystander is payoff of this mixed strategy prole: when she doesn't call for help, her expected payoff is $p^{n-1} \times 0 + (1 - p^{n-1}) \times v = (1 - p^{n-1})v$, and when she does call for help, her payoff is $v - c$. The fact that bystander i mixes her strategies implies that she is indifferent between these two, hence $(1 - p^{n-1})v = v - c$, and $p = (c/v)^{1/(n-1)}$. The probability that no one calls for help is $p^n = (c/v)^{n/(n-1)}$.

8. This is merely a reminder of basic matrix multiplication: the utilities are $\sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j u_1(s_i, t_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j A_{ij} = p^T A q$ and $\sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j u_2(s_i, t_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j B_{ij} = p^T B q$.

9. # Let $e_1, \ldots, e_n$ denote the standard column vectors. Notice that the Indifference Principle implies that the values of $p^T U_1 e_i$ are the same for all $1 \leq i \leq n$ and that the values of $e_j^T U_1 q$ are the same for all $1 \leq j \leq n$, i.e., there exist constants c_1, c_2 such that $p^T U_1 = c_1 e^T$ and $U_2 q = c_2 e$ where $e = e_1 + \cdots + e_n$. We can now write $p^T = c_1 e^T U_1^{-1}$ and $q = c_2 U_2^{-1} e$. To completely determine p and q add the condition that these are probability vectors: compute $p'^T = e^T U_1^{-1}$ and $q' = U_2^{-1} e^t$; now take $p = p'/(p'_1 + \cdots + p'_n)$ and $q = q'/(q'_1 + \cdots + q'_n)$. These are the only possible Nash equilibrium with all strategies in their support (and give a Nash equilibrium if and only if these p and q are probability vectors).

Alice's game-matrix A for this game is lower triangular with 1 on the diagonal, -1 below the diagonal and 0 above it. Bob's game-matrix is $-A$

Let (p, q) be an optimal strategy profile and assume that all pure strategies occur with positive probability. The Indifference Principle implies

$$\begin{cases} q_1 & = & V_1 \\ -q_1 + q_2 & = & V_1 \\ -q_1 - q_2 + q_3 & = & V_1 \\ & \vdots & \\ -q_1 - q_2 - q_3 - \ldots q_{n-1} + q_n & = & V_1 \end{cases} \begin{cases} p_1 - p_2 - \cdots - p_n & = & V_2 \\ p_2 - p_3 - \cdots - p_n & = & V_2 \\ p_3 - p_4 - \cdots - p_n & = & V_2 \\ & \vdots & \\ p_n & = & V_2 \end{cases}$$

where V_1 and V_2 are constants. The solutions of these are $V_1 = V_2 = 2/n(n+1)$, $q = (V_1, 2V_1, \ldots, nV_1)$ and $p = (nV_2, (n-1)V_2, \ldots, V_2)$. We note that p and q are indeed probability vectors.

10. **(a)** and **(b)** The mixed strategy $(1/2, 1/2, 0)$ and the pure strategy C guarantee the row and column players, respectively, an expected payoff of 0.

    (c) If $((p_1, p_2, 1-p_1-p_2), (q_1, q_2, 1-q_1-q_2))$ is a Nash equilibrium with $0 < p_1, p_2, q_1, q_2 < 1$, the principle of indifference implies that $-p_1+p_2-\alpha(1-p_1-p_2) = p_1-p_2-\alpha(1-p_1-p_2) = \delta(1-p_1-p_2)$. Solving this gives $-\alpha = \delta$ which is impossible as $\alpha, \delta > 0$.

    (d) If in equilibrium the row player eliminates I, the column player's C dominates B, hence B would not be played with positive probability. If $((0, p, 1-p), (q, 0, 1-q))$ is a Nash equilibrium, the principle of indifference implies that the row player's expected utility is $-q = \alpha q - \delta(1-q)$. If $q > 0$ the row player is better off switching to the strategy that guarantees a payoff of 0. If $q = 0$, $\alpha q - \delta(1-q) = -\delta$ and, again, there is a better response for the row player.

    (e) The indifference principle implies $-p + (1-p) = q - (1-p) = 0$ and $q_1 - q_2 = q_2 - q_1$, and we deduce that $p = 1/2$ and $q_1 = q_2$. Write q for both q_1 and q_2.

    The fact that this a Nash equilibrium means that $(1/2, 1/2, 0)$ is a best response against $(q, q, 1-2q)$, in particular, the row player does not mix strategy III because it must yield a lower utility against $(q, q, 1-2q)$, i.e., $0 > q\alpha + q\alpha - (1-2q)\delta$, hence $q < \delta/2(\alpha+\delta)$.

    (f) Such a Nash equilibrium is $((1/2, 1/2, 0), (0, 0, 1))$ with resulting utilities of 0 for both. To chech that this is indeed a Nash equilibrium, one needs to check that III versus C gives a negative utility for the row player, and that both A and B played against the row player's $(1/2, 1/2, 0)$ yield zero utility.

11. The expected payoffs are

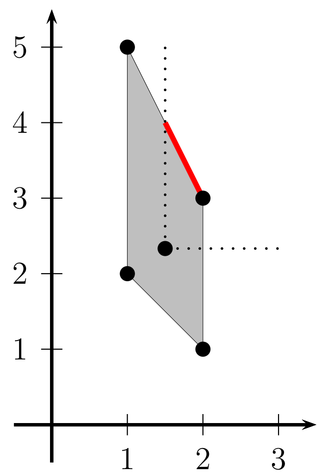
$$\begin{pmatrix} 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = 3/2$$

for Alice and

$$\begin{pmatrix} 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = 7/3$$

for Bob.

The cooperative payoff region is



where the red line segment is the subset of payoffs that satisfy the Individual Rationality and Pareto Optimality conditions. That line segment can be parametrized as $t(3/2, 4) + (1-t)(2, 3) = (2 - t/2, t + 3)$ for $0 \le t \le 1$.

To find the Nash bargain we maximize $(2 - t/2 - 3/2)(t + 3 - 7/3) = (1-t)(t + 2/3)/2$: ~~the maximum occurs at $t = 0$ and this corresponds to payoffs $(2, 3)$.~~ The critical point of this quadratic occurs at $t = 1/6$, and the values at $t = 0, 1/6$ and 1 are $12/36, 25/36$ and 0, so its maximum occurs at $t = 1/6$ corresponding to the point $(23/12, 19/6)$.