MAS348 Game Theory
Problem Sheet #2

1. Find all Nash equilibria of the following game in normal form

	l	r
U	1, 2	2, 1
D	2, 3	1, 5

2. Alice is kicking a penalty against goalkeeper Bob. She can choose to kick to the right or to the left of the goal and Bob can either jump to the right or left. The probabilities of Alice scoring a goal in these different scenarios is given by the following table:

	l	r
L	0	1
R	3/4	0

Describe all Nash equilibria of this game.

Repeat this with these probabilities:

	l	r
L	0.8	1
R	α	0.8

Note that Alice kicks to the left better than to the right; should she kick the the left more often? (You can read about a study of real-life football penalties in http://www.palacios-huerta.com/docs/professionals.pdf by Ignacio Palacios-Huerta.)

3. Explain why mixed-strategy Nash-equilibria do not include dominated strategies in their support.

4. # Consider the game $(\{1,\ldots,n\},\{1,\ldots,n\},u_1,u_2)$, where the $n \times n$ matrices $U_1 = (u_1(i,j))$ and $U_2 = (u_2(i,j))$ are invertible. Show that there exists at most one mixed-strategy Nash equilibrium (p,q) for which the support of both p and q is $\{1,\ldots,n\}$.

5. # Let N be a positive integer. Consider a game where two players bid a non-negative integer number of pounds less than N. If the bids are $x < y$, the person who bid y gets £$N - y$ and the other has to pay £x. If bids are (x,x), both players pay x. It is known that there exists a symmetric mixed Nash equilibrium $(p_0, p_1, \ldots, p_{N-1})$.

   (a) Describe this game in tabular form.

   (b) Deduce from the existence of mixed NE a linear system of equations satisfied by $(p_0, p_1, \ldots, p_{N-1})$.

(c) Find a solution to the system of equations in (b) (Hint: write the system for small N, say $N = 5$ and observe a simple looking solution. Guess the solution for general N and verify your guess.)

(d) Describe the mixed strategy NE.

6. A group of $n > 1$ people witness a pedestrian being hit by a car, and each has a mobile phone. The injured pedestrian is unconscious and requires immediate medical attention, which will be forthcoming if at least one person calls for help. Simultaneously and independently each of the n bystanders decides whether to call for help or not. Each bystander obtains v units of utility if the injured person receives help. Those who call pay a personal cost of $c < v$. If no one calls, each bystander receives a utility of 0. Find a symmetric Nash equilibrium of this game, i.e., a Nash equilibrium in which all players adopt the same strategy. What is the probability that no one calls for help in equilibrium?

7. Consider the zero-sum game with payoff matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 37 \end{bmatrix}.$$

Find all pure-strategy Nash equilibria of this game.

8. Consider the following zero-sum game

	A	B	C
I	1	-1	1
II	-1	0	1
III	2	-1	0

(a) Is there a saddle point in this game?

(b) What are the expected payoffs of the mixed strategy profile $((1/5, 2/5, 2/5), (1/3, 1/3, 1/3)$?

(c) If the row player is committed to playing $(1/5, 2/5, 2/5)$ what is the column player best strategy?

9. Consider the following zero-sum game

	A	B	C
I	-5	-1	2
II	-1	0	-1
III	1	-2	0

If the column player plays the mixed strategy $(1/5, 2/5, 2/5)$, which strategy should the row player play?

10. Suppose that a zero-sum game (S, T, u) has saddle points (s_1, t_1) and (s_2, t_2). Show that (s_1, t_2) is also a saddle point.

11. Consider the following zero-sum game

	A	B	C
I	1	-1	0
II	-1	1	0
III	α	α	$-\delta$

for some $\alpha, \delta > 0$.

(a) Find a strategy for the row player that guarantees her an expected payoff of 0.

(b) Find a strategy for the column player that guarantees him an expected payoff of 0.

(c) Find the value of the game.

(d) Find all Nash equilibria of the game.

12. Consider the following zero-sum game

	A	B	C
I	1	-1	2
II	-2	0	1
III	0	2	-1

(a) Are there any dominated strategies in this game?

(b) Are there any pure-strategy Nash equilibria in this game?

(c) Identify a mixed strategy profile (p^*, q^*) with $p^*(II) = 0$ and $q^*(C) = 0$ which is a candidate for being an optimal strategy profile.

(d) Verify that the strategy profile you found in (c) is indeed optimal.

13. Let A be the matrix associated with a symmetric game $G = (\{1, \ldots, n\}, \{1, \ldots, n\}, u)$. Assume further that $A = (a_{ij})$ has the property $a_{n-i+1,n-j+1} = -a_{i,j}$. Let

$$\sigma = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ & & \vdots & & \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

with 1 on its antidiagonal and 0 elsewhere. Show that $\sigma^T A \sigma = -A$ and deduce that if $(p_1, \ldots, p_n)$ is an optimal mixed strategy, then so is $(p_n, \ldots, p_1)$ and hence there exists an optimal mixed strategy $(q_1, \ldots, q_n)$ such that $q_1 = q_n, q_2 = q_{n-1}, \ldots$.

14. Two players choose simultaneously integers between 1 and 100, inclusive. If the numbers are equal, they both get nothing. A player who chooses a number just below the other player's pays £1 to the other player. A player who chooses a number 2 or more above the other player's pays £2 to the other player.

    (a) Describe the matrix for this game, explain why this game is symmetric.

    (b) By repeated elimination of dominated strategies, reduce this game to a smaller one.

    (c) Describe an optimal strategy profile for this game (Exercise 13 might be useful).

15. # Bob chooses a number from 1 to n, and Alice tries to guess his choice. If she guesses correctly, Bob pays her £1; if she guesses too low, she pays Bob £1; if she guesses too high there is no payoff.

    (a) What is the matrix of this game?

    (b) Find the value of the game and an optimal strategy profile under the assumption that the optimal strategies mix all pure strategies with positive probability.

    (c) Explain why the strategy profile you found in (b) is indeed optimal.