# MAS348 Game Theory
# Problem Sheet #2

1. Find all Nash equilibria of the following game in normal form

| | l | r |
|---|---|---|
| U | 1, 2 | 2, 1 |
| D | 2, 3 | 1, 5 |

2. The game of chicken is played by two drivers approaching narrow bridge from opposite directions. The first to swerve away yields the bridge to the other. If both drivers swerve, their utility is 0, and if one swerves and the other does not, the swerving driver's utility is -1, and the other driver's utility is 1. If neither driver swerves, the result is deadlock or a collision, with utility of -10 for both. Find all Nash equilibria of this game.

3. Alice is kicking a penalty against goalkeeper Bob. She can choose to kick to the right or to the left of the goal and Bob can either jump to the right or left. The probabilities of Alice scoring a goal in these different scenarios is given by the following table:

| | l | r |
|---|---|---|
| L | 0 | 1 |
| R | 3/4 | 0 |

   Describe all Nash equilibria of this game.

   Repeat this with these probabilities:

| | l | r |
|---|---|---|
| L | 0.8 | 1 |
| R | $\alpha$ | 0.8 |

   Note that Alice kicks to the left better than to the right; should she kick the the left more often? (You can read about a study of real-life football penalties in http://www.palacios-huerta.com/docs/professionals.pdf by Ignacio Palacios-Huerta.)

4. Explain why mixed-strategy Nash-equilibria do not include dominated strategies in their support.

5. Let $G = (S, T, u, v)$ be a finite game and consider the game $\cancel{\overline{G} = (S, T, v, w)}$ $\overline{G} = (S, T, u, w)$ where for all $s \in S$ and $t \in T$, $w(s, t) = av(s, t) + b$ for constants $a > 0$ and $b$. Show that a mixed strategy profile $(p, q)$ is a Nash equilibrium in $G$ if and only if it is a Nash equilibrium in $\overline{G}$.

6. # Let $N$ be a positive integer. Consider a game where two players bid a non-negative integer number of pounds less than N. If the bids are $x < y$, the person who bid $y$ gets $\pounds N - y$ and the other has to pay $\pounds x$. If bids are $(x, x)$, both players pay $x$. It is known that there exists a symmetric mixed Nash equilibrium $(p_0, p_1, \dots, p_{N-1})$.

   (a) Describe this game in tabular form.

   (b) Deduce from the existence of mixed NE a linear system of equations satisfied by $(p_0, p_1, \dots, p_{N-1})$.

   (c) Find a solution to the system of equations in (b) (Hint: write the system for small $N$, say $N = 5$ and observe a simple looking solution. Guess the solution for general $N$ and verify your guess.)

   (d) Describe the mixed strategy NE.

7. A group of $n > 1$ people witness a pedestrian being hit by a car, and each has a mobile phone. The injured pedestrian is unconscious and requires immediate medical attention, which will be forthcoming if at least one person calls for help. Simultaneously and independently each of the $n$ bystanders decides whether to call for help or not. Each bystander obtains $v$ units of utility if the injured person receives help. Those who call pay a personal cost of $c < v$. If no one calls, each bystander receives a utility of 0. Find a symmetric Nash equilibrium of this game, i.e., a Nash equilibrium in which all players adopt the same strategy. What is the probability that no one calls for help in equilibrium?

8. Consider a two-player game in normal form $(\{s_1, \dots, s_m\}, \{t_1, \dots, t_n\}, u_1, u_2)$. Let $p = (p_1, \dots, p_m)^T$ and $q = (q_1, \dots, q_n)^T$ be mixed strategies for both players (where $^T$ denotes transpose): the first player plays $s_i$ with probability $p_i$ and the second player plays $t_j$ with probability $q_j$. Let $A$ and $B$ be $m \times n$ matrices whose $i, j$ entries are $u_1(s_i, t_j)$ and $u_2(s_i, t_j)$, respectively, for $1 \le i \le m$ and $1 \le j \le n$. We shall call these the game-matrices for players 1 and 2, respectively, Show that if $p$ and $q$ are played, the (expected) utilities for both players are $p^T Aq$ and $p^T Bq$, respectively.

9. # Consider the game $(\{1, \dots, n\}, \{1, \dots, n\}, u_1, u_2)$, where the $n \times n$ game-matrices $U_1 = (u_1(i, j))$ and $U_2 = (u_2(i, j))$ are invertible. Show that there exists at most one mixed-strategy Nash equilibrium $(p, q)$ for which the support of both $p$ and $q$ is $\{1, \dots, n\}$.

   Bob chooses a number from 1 to $n$, and Alice tries to guess his choice. If she guesses correctly, Bob pays her £1; if she guesses too low, she pays Bob £1; if she guesses too high there is no payoff. Find a Nash equilibrium in which all strategies are mixed with positive probability.

10. Consider the following game

|  | A | B | C |
|---|---|---|---|
| I | 1, -1 | -1, 1 | 0, 0 |
| II | -1, 1 | 1, -1 | 0, 0 |
| III | $\alpha$, $-\alpha$ | $\alpha$, $-\alpha$ | $-\delta$, $\delta$ |

for some $\alpha, \delta > 0$.

(a) Find a strategy for the row player that guarantees her an expected payoff of 0.

(b) Find a strategy for the column player that guarantees him an expected payoff of 0.

(c) Show that there is no Nash equilibrium of the game in which the both players mix all their strategies with positive probability.

(d) Show that if in a Nash equilibrium the row player does not play I with positive probability, the column player will not mix strategy B. Deduce that there is no such Nash equilibrium.

(e) Assume we know that there are Nash equlibria of the form $((p, 1-p, 0), (q_1, q_2, q_3))$ with $0 < p, q_1, q_2, q_3 < 1$. Find all such equilibria.

(f) Find a Nash equilibrium in which the column player plays a pure strategy.

11. Alice and Bob face the following game

| | l | r |
|---|---|---|
| U | 1, 2 | 2, 1 |
| D | 2, 3 | 1, 5 |

(which we have seen has mixed Nash equilibrium $((2/3, 1/3), (1/2, 1/2))$) and choose to bargain an outcome.

(a) Find the expected payoffs at the Nash equilibrium.

(b) Sketch the cooperative payoff region of the game.

(c) Given that, if negotiations fail, the payoffs are $(3/2, 7/3)$, which payoffs in (a) satisfy the Individual Rationality and Pareto Optimality conditions.

(d) Find the Nash Bargain of this setup.