MAS348 Game Theory
Solutions #3

1. Consider the following strategy: whenever white places a piece on square (i, j), black places his piece on $(2n - i + 1, j)$. Black cannot end up with a column of his pieces, and if he ends up with a row, white must have completed a white row in the prior move.

2. We show by induction on n that Alice can force victory if and only if n is not divisible by $k + 1$. If $1 \leq n \leq k$, Alice wins by taking n tokens. Assume that $n > k$ and that the claim holds for all $m < n$. Write $n = q(k + 1) + r$.

   If $0 < r < k+1$, let Alice start by taking r pieces. Now Bob moves when there are $q(k+1)$ pieces, any any move leaves a number m of pieces which is not divisible by $k + 1$. By induction, Alice wins all those subgames.

   If $r = 0$, after any of Alice's moves, Bob plays the game with a number of tokens not divisible by $k + 1$, and the induction hypothesis guarantees him victory.

3. After reading this question, the words *Backward Induction* should be flashing in your mind– if not, plase review the relevant material before attempting this question again.

   We use Backward Induction as follows. If only Eve remains, she'll take all 100 coins.

   If only David and Eve remain, David knows that Eve can get all coins so he will offer all coins to Eve.

   If only Charles, David and Eve remain, Charles knows that David knows that if Charles gets thrown overboard, David will receive nothing, so Charles can offer 99 coins to himself, 1 coin for David and none to Eve and secure David's support.

   Bob knows that David and Eve know that if he gets throw overboard, they will receive 1 and 0 coins, respectively. So Bob can ensure their support by giving them 2 and 1 coins, respectively, and keeping the rest for himself.

   Now Alice knows that everyone knows that if she is thrown overboard, Charles and Eve will receive 0 and 1 coins, respectively, so she can get their support if she offers them 1 and 2 coins, respectively, and takes 97 coins for herself.

4. Assume this is not the case; hence Zermelo's Theorem implies that the second player has a strategy which guarantees him victory. Alice now steals that strategy as follows.

   Alice starts by picking any random square S and henceforth plays the game as if she were Bob and as if S were empty. If, however, this strategy calls for taking square S, Alice makes a random move. If, later in the game, the strategy calls for taking a square already owned by Alice, she makes any random game. The game ends either with a winning position in Bob's winning strategy book (with Bob and Alice interchanged), or with such a wining position plus an extra square for Alice, which is also a winning position.

5. The first player takes the square diagonally adjacent to the poisoned square, resulting in an "L" shape bar. From that point on, the first player repeats whatever the second player plays, transposed to the other leg of the "L" shaped bar.

6. Backward induction gives the solution (U, L_1).

   The normal form of the game is

	$[L_1, L_2]$	$[L_1, M_2]$	$[L_1, D_2]$	$[M_1, L_2]$	$[M_1, M_2]$	$[M_1, R_2]$	$[R_1, L_2]$	$[R_1, M_2]$	$[R_1, R_2]$
U	5, 2	5, 2	5, 2	1, 1	1, 1	1, 1	3, 0	3, 0	3, 0
D	2, 1	3, 4	1, 0	2, 1	3, 4	1, 0	2, 1	3, 4	1, 0

   The pure strategy Nash-equilibria are: $(U, [L_1, L_2])$, $(U, [L_1, M_2])$, $(U, [L_1, R_2])$, $(D, [M_1, M_2])$, $(D, [R_1, M_2])$. Only $(U, [L_1, M_2])$ is subgame-perfect. (There is a mixed strategy too, but it is not easy to find it.)

7. If we perform backward induction, we will discover that Alice chooses E at node v_4 and so both choices of Bob at v_3 yield him a payoff of 2, and he might choose either. So Alice's choice of D might yield her either 0 or 6, while he choice of U will result in a payoff of 5. So it is not clear from this analysis what should Alice do at node v_1.

   To describe the normal form of the game, we note that Alice needs to decide U or D in node v_1 and E or F in node v_4, whereas Bob needs to choose between a and b in node v_2 and between c and d in node v_3. So each has 4 strategies; Alice's are $[U, E]$, $[U, F]$, $[D, E]$, $[D, F]$ and Bob's are $[a, c]$, $[a, d]$, $[b, c]$, $[b, d]$. We can now give the normal form of the game as

	$[a, c]$	$[a, d]$	$[b, c]$	$[b, d]$
$[U, E]$	2, 3	2, 3	5, 4	5, 4
$[U, F]$	2, 3	2, 3	5, 4	5, 4
$[D, E]$	6, 2	0, 2	6, 2	0, 2
$[D, F]$	2, 6	0, 2	2, 6	0, 2

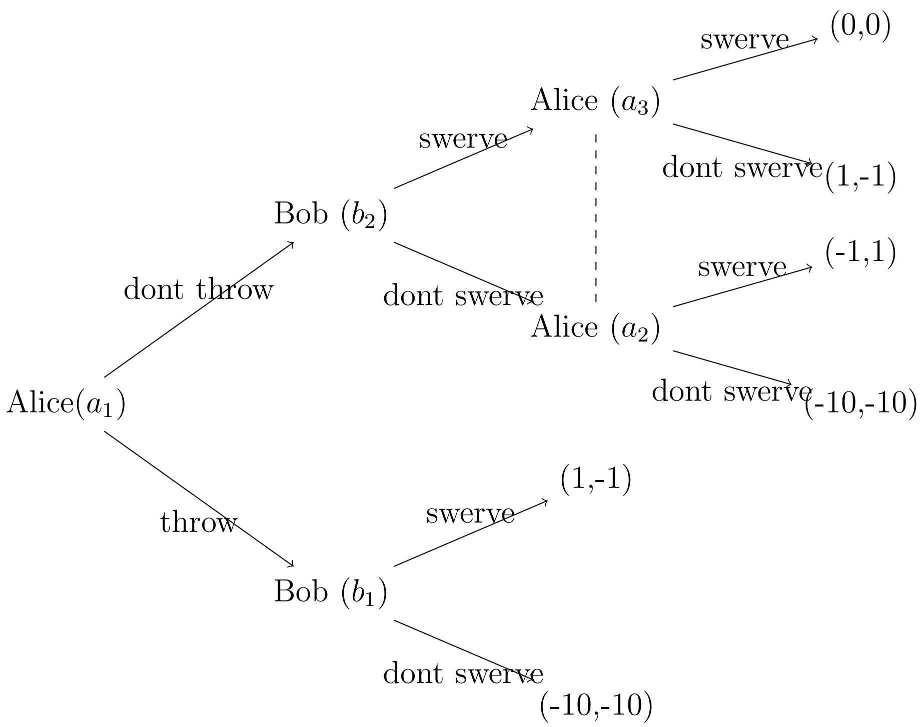
   To find Nash equilibria, we underline best responses

	$[a, c]$	$[a, d]$	$[b, c]$	$[b, d]$
$[U, E]$	2, 3	$\underline{2}$, 3	5, $\underline{4}$	$\underline{5}$, $\underline{4}$
$[U, F]$	2, 3	$\underline{2}$, 3	5, $\underline{4}$	$\underline{5}$, $\underline{4}$
$[D, E]$	$\underline{6}$, $\underline{2}$	0, $\underline{2}$	$\underline{6}$, $\underline{2}$	0, $\underline{2}$
$[D, F]$	2, $\underline{6}$	0, 2	2, $\underline{6}$	0, 2

   and find four Nash equilibria: $([U, E], [b, d])$, $([U, F], [b, d])$, $([D, E], [a, c])$, $([D, E], [b, c])$.

   $([U, F], [b, d])$ fails to restrict to a Nash equilibrium of the game starting at node v_4, and $([D, E], [a, c])$ fails to restrict to a Nash equilibrium of the game starting at node v_2. The other two are subgame perfect.

8. The tree associated with this game is as follows.



($\{a_2, a_3\}$ is an information set)

In normal form, Alice strategies describe her two decisions at node a_1 (*throw* or *dont throw*, abbreviated T, DT), and at the information set $\{a_2, a_3\}$ (*swerve* or *dont swerve* , abbreviated S, DS). So Alice has 4 strategies [T, S], [T, DS], [DT, S] and [DT, DS]. Bob makes decision at nodes b_1 and b_2, and his strategies are [S, S], [S, DS], [DS, S] and [DS, DS]. The game in normal tabular form is given by

	$[S, S]$	$[S, DS]$	$[DS, S]$	$[DS, DS]$
$[T, S]$	1,-1	1, -1	-10, -10	-10, -10
$[T, DS]$	1,-1	1, -1	-10, -10	-10, -10
$[DT, S]$	0, 0	-1, 1	0, 0	-1, 1
$[DT, DS]$	1, -1	-10, -10	1, -1	-10, -10

We indicate best responses

	$[S, S]$	$[S, DS]$	$[DS, S]$	$[DS, DS]$
$[T, S]$	$\underline{1}, \underline{-1}$	$\underline{1}, \underline{-1}$	-10, -10	-10, -10
$[T, DS]$	$\underline{1}, \underline{-1}$	$\underline{1}, \underline{-1}$	-10, -10	-10, -10
$[DT, S]$	0, 0	$-1, \underline{1}$	0, 0	$\underline{-1}, \underline{1}$
$[DT, DS]$	$\underline{1}, \underline{-1}$	-10, -10	$\underline{1}, \underline{-1}$	-10, -10

and discover 7 Nash equilibria. Among these, ([T, S], [S, S]) , ([T, DS], [S, DS]) , ([DT, S], [DS, S]) and ([DT, DS], [DS, DS]) are not subgame perfect.

If Alice decides at node a_1 not to throw away her steering wheel, she ends up playing the game as described in sheet 2, and her expected utility is

$$(0) \times 0.81 + (-1) \times 0.1 \times 0.9 + (1) \times 0.9 \times 0.1 + (-10) \times 0.1 \times 0.1 = -1/10$$

If she cooses to throw away her steering wheel, Bob will choose to swerve, and Alice's utility is 1. Hence Alice will throw away her steering wheel.