MAS348 Game Theory
Solutions #3

1. Consider the following strategy: whenever white places a piece on square (i, j), black places his piece on $(2n - i + 1, j)$. Black cannot end up with a column of his pieces, and if he ends up with a row, white must have completed a white row in the prior move.

2. We show by induction on n that Alice can force victory if and only if n is not divisible by $k + 1$. If $1 \leq n \leq k$, Alice wins by taking n tokens. Assume that $n > k$ and that the claim holds for all $m < n$. Write $n = q(k + 1) + r$.

   If $0 < r < k+1$, let Alice start by taking r pieces. Now Bob moves when there are $q(k+1)$ pieces, any any move leaves a number m of pieces which is not divisible by $k + 1$. By induction, Alice wins all those subgames.

   If $r = 0$, after any of Alice's moves, Bob plays the game with a number of tokens not divisible by $k + 1$, and the induction hypothesis guarantees him victory.

3. Assume this is not the case; hence Zermelo's Theorem implies that the second player has a strategy which guarantees him victory. Alice now steals that strategy as follows.

   Alice starts by picking any random square S and henceforth plays the game as if she were Bob and as if S were empty. If, however, this strategy calls for taking square S, Alice makes a random move. If, later in the game, the strategy calls for taking a square already owned by Alice, she makes any random game. The game ends either with a winning position in Bob's winning strategy book (with Bob and Alice interchanged), or with such a wining position plus an extra square for Alice, which is also a winning position.

4. Backward induction gives the solution (U, L_1).

   The normal form of the game is

	$[L_1, L_2]$	$[L_1, M_2]$	$[L_1, D_2]$	$[M_1, L_2]$	$[M_1, M_2]$	$[M_1, R_2]$	$[R_1, L_2]$	$[R_1, M_2]$	$[R_1, R_2]$
U	5, 2	5, 2	5, 2	1, 1	1, 1	1, 1	3, 0	3, 0	3, 0
D	2, 1	3, 4	1, 0	2, 1	3, 4	1, 0	2, 1	3, 4	1, 0

   The pure strategy Nash-equilibria are: $(U, [L_1, L_2])$, $(U, [L_1, M_2])$, $(U, [L_1, R_2])$, $(D, [M_1, M_2])$, $(D, [R_1, M_2])$. Only $(U, [L_1, M_2])$ is subgame-perfect. (There is a mixed strategy too, but it is not easy to find it.)

5. If we perform backward induction, we will discover that Alice chooses E at node v_4 and so both choices of Bob at v_3 yield him a payoff of 2, and he might choose either. So Alice's choice of D might yield her either 0 or 6, while he choice of U will result in a payoff of 5. So it is not clear from this analysis what should Alice do at node v_1.

   To describe the normal form of the game, we note that Alice needs to decide U or D in node v_1 and E or F in node v_4, whereas Bob needs to choose between a and b in node

v_2 and between c and d in node v_3. So each has 4 strategies; Alice's are $[U, E]$, $[U, F]$, $[D, E]$, $[D, F]$ and Bob's are $[a, c]$, $[a, d]$, $[b, c]$, $[b, d]$. We can now give the normal form of the game as

	$[a, c]$	$[a, d]$	$[b, c]$	$[b, d]$
$[U, E]$	2, 3	2, 3	5, 4	5, 4
$[U, F]$	2, 3	2, 3	5, 4	5, 4
$[D, E]$	6, 2	0, 2	6, 2	0, 2
$[D, F]$	2, 6	0, 2	2, 6	0, 2

To find Nash equilibria, we underline best responses

	$[a, c]$	$[a, d]$	$[b, c]$	$[b, d]$
$[U, E]$	2, 3	$\underline{2}$, 3	5, $\underline{4}$	$\underline{5}$, $\underline{4}$
$[U, F]$	2, 3	$\underline{2}$, 3	5, $\underline{4}$	$\underline{5}$, $\underline{4}$
$[D, E]$	$\underline{6}$, $\underline{2}$	0, $\underline{2}$	$\underline{6}$, $\underline{2}$	0, $\underline{2}$
$[D, F]$	2, $\underline{6}$	0, 2	2, $\underline{6}$	0, 2

and find four Nash equilibria: $([U, E], [b, d])$, $([U, F], [b, d])$, $([D, E], [a, c])$, $([D, E], [b, c])$.

$([U, F], [b, d])$ fails to restrict to a Nash equilibrium of the game starting at node v_4, and $([D, E], [a, c])$ fails to restrict to a Nash equilibrium of the game starting at node v_2. The other two are subgame perfect.