MAS348 Game Theory
Solutions #4

1. (a) Both strategy profiles result in both players playing DOVE in each repetition of Prisoners' Dilemma, hence the payoffs are $2 + 2\beta + 2\beta^2 + \cdots = 2/(1-\beta)$.

   (b) TIT-FOR-TAT is not a best response to HAWK, because HAWK fares better against HAWK. Similarly GRIM does less well against HAWK than HAWK.

   (c) (h,H) is a Nash equilibrium for each instance of Prisoners' Dilemma, so Theorem 1 implies that (HAWK,HAWK) is a Nash equilibrium.

   (d) HAWK fares better against DOVE than DOVE, so (DOVE, DOVE) is not a Nash equilibrium.

   (e) A player who deviates from GRIM at the kth stage of the game will receive payoff

   $$2(1 + \beta + \ldots \beta^{k-2}) + 3\beta^{k-1} = 2\frac{1-\beta^{k-1}}{1-\beta} + 3\beta^{k-1}$$

   so the player will not deviate as long as this does not exceed the payoff $2/(1-\beta)$ we found in part (a), hence β needs to satisfy

   $$2\frac{1-\beta^{k-1}}{1-\beta} + 3\beta^{k-1} \leq 2/(1-\beta)$$

   yielding $\beta \geq 1/3$.

   (f) If there is a profitable deviation from TIT-FOR-TAT that starts by playing h in the ith stage of the game, then, since the game starting at that stage is identical to the original game, there is also a profitable deviation starting with playing h in the 0th stage. Consider playing h in stages $0, \ldots, k-1$ and playing d on stage k (thus returning to the original game). The payoff from these $k+1$ stages is $3 - \beta^k$ compared to $2(1-\beta^{k+1})/(1-\beta)$ that we obtain playing TIT-FOR-TAT vs. TIT-FOR-TAT in the first $k+1$ stages, hence for (TIT-FOR-TAT, TIT-FOR-TAT) to be a Nash equilibrium β needs to satisfy $3 - \beta^k \leq 2(1-\beta^{k+1})/(1-\beta)$ which is equivalent to $(3\beta - 1)(\beta^k - 1) \leq 0$ and hence $\beta \geq 1/3$.

2. We checked that this is a (non-subgame perfect) Nash equilibrium when the game is played once, and so the strategy profile is a subgame perfect Nash equilibrium by Theorem 1.

3. (a) D dominates U and R dominates L. After eliminating dominated strategies we are left with one possible strategy profile (D,R) which must be then a Nash equilibrium.

   (b) Let $\mathcal{G}_1$ be the strategy for the row player in which, if player the column player ever deviated from L she plays D and she plays U as long as the column players sticks to L. Let $\mathcal{G}_2$ be the strategy for the column player in which, if the row player ever deviated from U he plays R and he plays L as long as the row player sticks to U.

If $(\mathcal{G}_1, \mathcal{G}_2)$ is played, both players get a payoff of $4\sum_{i=0}^{\infty}(99/100)^i = 400$, so it suffices to show that $(\mathcal{G}_1, \mathcal{G}_2)$ is a subgame-perfect Nash equilibrium for $G^\infty(99/100)$.

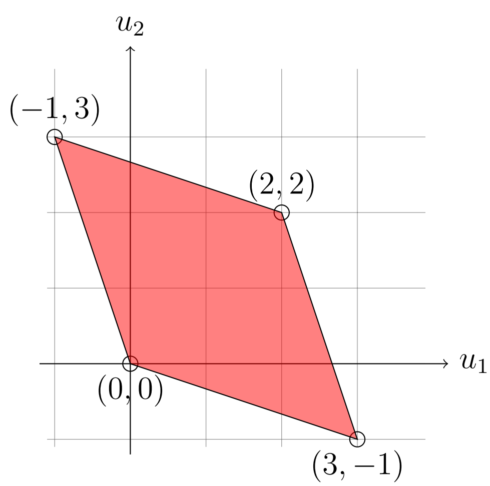
If the row player deviates from playing U by playing D at the k stage of the game and s_i $(i > k)$ thereafter, she get the payoff

$$\begin{aligned}
4\sum_{i=0}^{k-1}(99/100)^i + 50(99/100)^k + \sum_{i=k+1}^{\infty}(99/100)^i u_1(s_i, \mathrm{R}) &\leq 4\frac{1-(99/100)^k}{1-99/100} + 50(99/100)^k \\
&= (99/100)^k(-400+50) + 400 \\
&< 400
\end{aligned}$$

Similarly, if the column player deviates from playing L by playing R at the k stage of the game and t_i $(i > k)$ thereafter, she get the payoff

$$\begin{aligned}
4\sum_{i=0}^{k-1}(99/100)^i + 70(99/100)^k + \sum_{i=k+1}^{\infty}(99/100)^i u_2(\mathrm{D}, t_i) &\leq 4\frac{1-(99/100)^k}{1-99/100} + 70(99/100)^k \\
&= (99/100)^k(-400+70) + 400 \\
&< 400
\end{aligned}$$

4. (a)



(b) The minimax values are $0 = u_1(h, H)$ and $0 = u_2(h, H)$.

(c) Write $(5/3, 1/3) = 1/3(2,2) + 1/3(3,-1)$. Consider player 1's strategy s consisting of playing d, h, h cyclically and player 2's strategy t consisting of playing D, D, H cyclically. The average payoffs of the strategy profile (s, t) are $5/3$ for player 1 and $1/3$ for player 2.

Let $\mathcal{S}_1$ be player 1's strategy which consists of playing s as long as player 2 plays t, but to switch to playing h forever after player 2 deviates from t. Let $\mathcal{S}_2$ be player 2's strategy which consists of playing t as long as player 2 plays s, but to switch to playing h forever after player 1 deviates from s.

If player 1 deviates from $\mathcal{S}_1$, she will end up with an average payoff of $0 < 5/3$, and if player 2 deviates from $\mathcal{S}_1$, she will end up with an average payoff of $0 < 1/3$.

(d) Yes. Any subgame of the game before a defection is identical to the original game, so if a subgame is played before a defection, $(\mathcal{S}_1, \mathcal{S}_2)$ is a Nash equilibrium of that subgame. If either player defects, in the subsequent subgames strategies $\mathcal{S}_1$ and $\mathcal{S}_2$ dictate playing a Nash equilibrium (h, h) in each repetition of Prisoner's dilemma, which we have shown to be a Nash equilibrium.

5. # (a) The point is that $x - m_1 > 0$ and $y - m_2 > 0$ so we can take C to be any integer bigger than $N \max_{s \in S, t \in T}(u_1(s,t) - x)/(x - m_1)$ and $N \max_{s \in S, t \in T}(u_2(s,t) - y)/(y - m_2)$.

(b) If player 1 defects during a N-cycle, the total gains over the cycle compared with the payoffs of $\mathcal{G}_1$ are at most $N \max_{s \in S, t \in T}(u_1(s,t) - x)$ and the total loss over the next C-cycle compared with the payoffs of $\mathcal{G}_1$ are at least $C(x - m_1)$ resulting in a negative deviation from the average x.

Similarly, one shows that player 2 looses from deviating from $(\mathcal{G}_1, \mathcal{G}_2)$.

(c) To show that $(\mathcal{G}_1, \mathcal{G}_2)$ is a *subgame perfect* Nash equilibrium for G^∞, we need to show that if in a subgame, say, player 1 should be retaliating (i.e., playing σ_2) there is no incentive no to do so. The point is that after retaliation is finished, the game reverts to the Nash equilibrium $(\mathcal{G}_1, \mathcal{G}_2)$ and the losses incurred by retaliation do not affect the average payoff as the limit of the numbers of repetitions of the game go to ∞.

6. (a) The normal form of this sequential game is

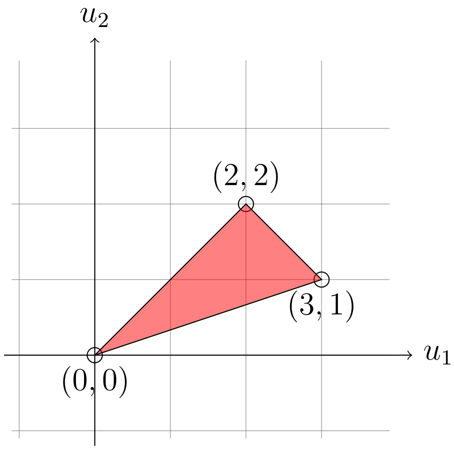
	accept 2 accept 1	accept 2 decline 1	decline 2 accept 1	decline 2 decline 1
offer 2	(2,2)	(2,2)	(0,0)	(0,0)
offer 1	(3,1)	(0,0)	(3,1)	(0,0)

There are three Nash equilibria: (offer 2, [accept 2, decline 1]), (offer 1, [accept 2, accept 1]) and (offer 1, [decline 2, accept 1]).

(b) If Bob has to make a decision, it would be disadvantageous to decline any offer, hence only (offer 1, [accept 2, accept 1]) is subgame perfect.

(c) The minimax values of the payoffs for Alice and Bob are 0 and 1, respectively.

(d)



(e) Since $(2,1) = (1/4)(2,2) + (1/2)(3,1) + (1/4)(0,0)$ the strategy profile (s,t) consisting of

- Alice playing cyclically offer 2, offer 1, offer 1, offer 1, and
- Bob playing cyclically *accept all*, *accept all*, *accept all*, *decline all*

yields a payoff of (2,1).

(f) The Nash equilibrium strategy profile is to play (s,t) above as long as the other player sticks to it, for Alice to play *unequal* if Bob deviates from t and for Bob to play *decline all* if Alice deviates from s.

7. We model this as Bayesian game with Alice having two types (Fit and Not Fit) and Bob having one type, we have two states of the world, in the first Alice is Fit and the secons she is Not Fit. Alice has 4 strategies: [Fit→ fight, Not Fit → fight], [Fit→ fight, Not Fit → dont fight], [Fit→ dont fight, Not Fit→ fight], [Fit→ dont fight, Not Fit→ dont fight], which we abbreviate [f, f], [f, d], [d, f], [d, d]. We obtain the following payoffs in tabular form

	fight	dont fight
$[f,f]$	$([1,-2],-3p+1)$	$([2,2],-1)$
$[f,d]$	$([1,-1],-4p+2)$	$([2,0],-p)$
$[d,f]$	$([-1,-2],p+1)$	$([0,2],p-1)$
$[d,d]$	$([-1,-1],2)$	$([0,0],0)$

Alice's best response to Bob's fight is $[f,d]$ giving Bob expected payoff $-4p+2$. Alice's best response to Bob's dont fight is $[f,f]$, giving Bob expected payoff -1. So he will fight when $-4p+2 > -1$, i.e., when $p < 3/4$.

8. Elimination of Alice's dominated strategies reveals that Alice always attacks: Bob's expected payoff is $-25p-(1-p)$ if he attacks and $-20p-3(1-p)$ if he doesn't, so he attacks when $-25p-(1-p) \geq -20p-3(1-p)$, i.e., when $p \leq 2/7$.

9. There is one type for Alice, and two for Bob: call them Y (likes) and N (doesn't like). There are two states of the world ω_1, ω_2.

A strategy for Alice is an element in $\{R\&C, S\&N\}$; a strategy for Bob is a function from $\{Y, N\}$ to $\{R\&C, S\&N\}$ and there are 4 of these: $[Y \mapsto R\&C, N \mapsto R\&C]$, $[Y \mapsto R\&C, N \mapsto S\&N]$, $[Y \mapsto S\&N, N \mapsto R\&C]$ and $[Y \mapsto S\&N, N \mapsto S\&N]$. We abbreviate these $[R\&C, R\&C]$, $[R\&C, S\&N]$, $[S\&N, R\&C]$ and $[S\&N, S\&N]$. The expected payoffs of these strategy profiles are as follows

	[R&C, R&C]	[R&C, S&N]	[S&N, R&C]	[S&N, S&N]
R&C	2, [1, 0]	1, [1, 2]	1, [0, 0]	0, [0, 2]
S&N	0, [0, 1]	1/2, [0, 0]	1/2, [2, 1]	1, [2, 0]

We now indicate the best responses for Alice *and for each type of Bob* as follows

	[R&C, R&C]	[R&C, S&N]	[S&N, R&C]	[S&N, S&N]
R&C	$\underline{2}$, [$\underline{1}$, 0]	$\underline{1}$, [$\underline{1}$, $\underline{2}$]	$\underline{1}$, [0, 0]	0, [0, $\underline{2}$]
S&N	0, [0, $\underline{1}$]	1/2, [0, 0]	1/2, [$\underline{2}$, $\underline{1}$]	$\underline{1}$, [$\underline{2}$, 0]

and we get a Bayes-Nash equilibrium at the strategy profile (R&C, [R&C, S&N]).

10. (a) The set of actions for Alice is the set of integers in $[0, 9]$, and Bob has two actions available: Accept and Reject. Alice has one type: Alice; Bob has 10 types $B_0, \ldots, B_9$ where Bob has type B_i when the value of his firm is i million pounds. Ω is the set of integers in $[0, 9]$, $\tau_A(i) = \text{Alice}$, $\tau_B(i) = B_i$. Alice has ten strategies: her ten possible bids. Bob's strategies are the sets of functions from $\{B_0, \ldots, B_9\}$ to $\{\text{Accept}, \text{Reject}\}$. $\text{Prob}(i|\text{Alice}) = 1/10$; $\text{Prob}(i|B_j)$ is 0 if $i \neq j$ and 1 if $i = j$.

    (b) Bob accepts a bid k iff $k > n$, so Alice's strategy of bidding k has expected payoff

$$\sum_{n=0}^{k-1} \frac{1}{10} \left(\frac{3}{2} n - k \right) = \frac{1}{10} \left(\frac{3}{2} \frac{(k-1)k}{2} - k^2 \right) = -\frac{1}{40}(k^2 + 3k) \leq 0;$$

    this is negative unless $k = 0$, so the only Bayes-Nash equilibrium occurs when Alice bids 0 and all types of Bob reject it.

    (c) Alice's bid is accepted only when it is too high on average, so positive bids do not survive competition in the environment of this example.