MAS348 Game Theory
Problem Sheet #4

1. Consider the following version of Prisoners' Dilemma

	D	H
d	2, 2	-1, 3
h	3, -1	0, 0

We call d and D dove strategies and h and H hawk strategies. This game is repeated indefinitely; before each repetition the game is stopped with probability $1-\beta$ for some $0<\beta<1$. Consider the following strategies available to both players:

HAWK Always play hawk.

DOVE Always play dove.

GRIM Play dove until the other player plays hawk; after that always play hawk.

TIT-FOR-TAT Start by playing dove. After that play dove if the other played dove last time, or play hawk otherwise.

(a) Compute the payoffs of the strategy profiles (TIT-FOR-TAT, TIT-FOR-TAT) and (GRIM, GRIM)

(b) Show that neither (HAWK, TIT-FOR-TAT) nor (HAWK, GRIM) are Nash equilibria.

(c) Show that (HAWK,HAWK) is a Nash equilibrium.

(d) Show that (DOVE, DOVE) is not a Nash equilibrium.

(e) For which values of β is (GRIM, GRIM) a Nash equilibrium?

(f) $^{\#}$ For which values of β is (TIT-FOR-TAT, TIT-FOR-TAT) a Nash equilibrium?

2. Recall our example in which Bob has a monopoly in a market of wireless widgets, and Alice is considering competing with Bob. If Alice does not enter the market, her payoff is zero, and Bob makes a three billion pound profit. If Alice enters the market, Bob can either

- fight her off by reducing prices thus making zero profit and causing Alice to lose a billion pounds, or
- not fight her off and they both end up making a profit of a billion pounds.

Consider the strategy profile in the repeated game $G^{\infty}(\beta)$ consisting of Alice not entering the market and Bob fighting her off. Explain why is this a subgame perfect Nash equilibrium.

3. Consider the game G given in normal form as follows

	L	R
U	4, 4	-1, 70
D	50, -1	0, 0

(a) Identify all dominated strategies and all Nash equilibria of G.

(b) Describe a Nash equilibrium of $G^\infty(99/100)$ which results in an expected payoff of $4/(1 - 99/100) = 400$ for both players. Justify your answer in detail.

(c) If we were to change $u_1(\mathrm{D}, \mathrm{L})$ from 50 to 500, would there be any Nash equilibrium which results in an expected payoff of at least 400 for the column player.

4. Consider the following version of Prisoners' Dilemma

	D	H
d	2, 2	-1, 3
h	3, -1	0, 0

(a) Sketch the cooperative payoff region of the game.

(b) Find the minimax values of each player.

(c) Exhibit a Nash equilibrium in which the average reward payoffs of the infinitely repeated game are 5/3 and 1/3 for the row and column players, respectively. Justify your answer fully.

(d) Is your Nash equilibrium in (c) subgame perfect?

5. # In this exercise we show that the dense set of payoffs in the second version of the "folk theorem" consist of payoffs of subgame perfect Nash equilibria.

Let $G = (S, T, u_1, u_2)$ be a finite game with minimax values $m_1 = u_1(\sigma_1, \tau_1)$ and $m_2 = u_1(\sigma_1, \tau_1)$. Let R be the cooperative payoff region of G and let $A = \{(x, y) \in R \,|\, x > m_1, y > m_2\}$.

Consider $(x, y) \in A$ of the form $(x, y) = \lambda_1(u_1(s_1, t_1), u_2(s_1, t_2)) + \cdots + \lambda_k(u_1(s_k, t_k), u_2(s_k, t_k))$ where $k \geq 1$, $s_1, \ldots, s_k \in S$, $t_1, \ldots, t_k \in T$ and $\lambda_1, \ldots \lambda_k$ are non-negative *rational* numbers in $[0, 1]$ adding up to 1, Also, we an find a positive integer N such that $\lambda_1 = m_1/N, \ldots, \lambda_k = m_k/N$ for integers $m_1, \ldots, m_k$. We will show that there exists a subgame perfect Nash equilibrium of G^∞ whose average payoff is (x, y).

(a) Explain why is there a large integer C such that $N \max_{s \in S, t \in T}(u_1(s, t) - x) < C(x - m_1)$ and $N \max_{s \in S, t \in T}(u_2(s, t) - y) < C(y - m_2)$.

(b) Let s be the strategy for player 1 which consists of playing s_1 for m_1 turns followed by s_2 for m_2 turns, etc., ending with playing s_k for m_k turns and repeating this pattern cyclically. Let t be the strategy for player 2 which consists of playing t_1 for m_1 turns followed by t_2 for m_2 turns, etc., ending with playing t_k for m_k turns and repeating this pattern cyclically. Let $\mathcal{G}_1$ be the strategy for player 1 in which s is played as long as player 2 plays t, but if player 2 defects, player 1 plays σ_2 C times, after which she resumes playing s. Let $\mathcal{G}_2$ be the strategy for player 2 in which t is played as long as player 1 plays 2, but if player 1 defects, player 2 plays τ_1 C times, after which he resumes playing t. Show that $(\mathcal{G}_1, \mathcal{G}_2)$ is a Nash equilibrium for G^∞.

(c) Show that $(\mathcal{G}_1, \mathcal{G}_2)$ is a *subgame perfect* Nash equilibrium for G^∞.

6. (The Ultimatum Game) Consider the following game G: Alice can either share £4 equally with Bob or to offer Bob £1 and to keep £3. Bob can either accept the offer, or he can decline it, in which case they get nothing.

    (a) Find all Nash equilibria of this game.

    (b) Find all subgame perfect equilibria of this game.

    (c) Find the players' minimax values of their payoffs.

    (d) Find the cooperative payoff region of the game.

    (e) Exhibit a strategy profile of G^∞ which results in average payoff $(2, 1)$.

    (f) Exhibit a Nash equilibrium of G^∞ which results in average payoff $(2, 1)$.

7. Alice and Bob prepare to fight and there two cases: either Alice has been working out in they gym and she is now fit, or she isnt fit. If Alice is fit, they conflict is modeled by the first game below, and if she isnt, they play the second game.

Alice is fit

	fight	dont fight
fight	1, -2	2, -1
dont fight	-1, 2	0, 0

Alice is not fit

	fight	dont fight
fight	-2, 1	2, -1
dont fight	-1, 2	0, 0

Bob knows that Alice is fit with probability p. For what values of p should Bob fight?

8. Alice and Bob are on brink of war, and each can either attack the other or not. Alice either has or hasn't nuclear weapons and all Bob knows are the probabilities p and $1 - p$ of these cases, respectively.

Alice has nucleal weapons

	attack	dont attack
attack	4, -25	5, -20
dont attack	0, 5	0, 0

Alice has no nucleal weapons

	attack	dont attack
attack	-1, -1	5, -3
dont attack	-3, 5	0, 0

For what values of p should Bob attack?

9. Alice and Bob meet in a lecture and decide to to on a date that night to one of their two favourite pubs (the "Rabbit and Carrot" and the "Squirrel and Nut") but they forgot to specify which one. Alice likes Bob, but Alice thinks that Bob likes her with probability 1/2. On that evening they need to chose where to go. Their payoffs, depending on whether Bob likes Alice are as follows.

Bob likes Alice

	R&C	S&N
R&C	2, 1	0, 0
S&N	0, 0	1, 2

Bob doesn't like Alice

	R&C	S&N
R&C	2, 0	0, 2
S&N	0, 1	1, 0

Find the Bayes-Nash equilibria for this game.

10. Alice wants to buy a firm owned by Bob. The value of the firm is n million pounds for some integer $0 \leq n \leq 9$; the value of the firm is known to Bob, but not to Alice who assigns equal probability to each one of the ten valuations of the firm. In addition, Alice knows that the firm under her management would be worth 50% more then its worth when owned by Bob. Alice will make a bid of k million pounds for some integer $0 \leq k \leq 9$.

    (a) Model this as a Bayesian game.

    (b) Find all Bayes-Nash equilibria of this game. (Assume that Bob accepts a bid of k iff $k > n$.)

    (c) This type of game is described by game theorists as a *game of adverse selection.* Can you think why?