MAS362 – 2012-13 Exam Solutions

1(i) (a) $Y(0.5) = -\log 0.9925/0.5 \approx 1.5\%$.
(b) To find $Y(1)$ we solve

$$3 \times 0.9925 + 103e^{-Y(1)\times 1} = 103.94$$

for the unknown $Y(1)$. We have $Y(1) \approx -\log\left((103.94 - 3 \times 0.9925)/103\right) \approx 2\%$.
To find $Y(1.5)$ we solve

$$5 \times 0.9925 + 5e^{-Y(1)\times 1} + 105e^{-Y(1.5)\times 1.5} = 111$$

for the unknown $Y(1.5)$.
We have

$$Y(1.5) \approx 2.5\%.$$

(b) The forward rate for deposits from 1 to 1.5 years is $(1.5\times 2.5\% - 1\times 2\%)/0.5 = 3.5\% > 2.5\%$, so we enter the forward agreement as a borrower.

1. We borrow $£1,000e^{-0.02}$ for 1 year and
2. deposit $£1,000e^{-0.02}$ for 1.5 years.
3. After 1 year we borrow $£1,000$ for 6 months at an interest rate of 2.5% and
4. use it to repay our loan.
5. After an additional 6 months we obtain balance of the deposit which amounts to

$$£1,000e^{-0.02}e^{0.025\times 1.5} \approx 1,017.654$$

and

6. repay the balance of our loan which amounts to $£1,000e^{0.025\times 0.5} \approx 1,012.578$ and
7. we pocket the difference of $£1,017.654 - 1,012.578 = 5.07 > 0$.

(ii)(a) The present value of the dividend is £$0.5e^{-0.02\times 0.5} \approx 0.495$ and the forward price is $(10 - 0.5e^{-0.02\times 0.5})e^{0.02} \approx 9.70$.
(ii)(b)

1. take a short position in the forward contract,
2. borrow £$0.5e^{-0.02\times 0.5}$ for six months at a spot interest rate of 2%,
3. borrow £$10 - 0.5e^{-0.02\times 0.5}$ for 1 year at a spot interest rate of 2%,
4. buy the asset for £10,
5. wait six months,
6. collect the 50p dividend and use it to repay first loan, now amounting to 50p,
7. wait another six months,
8. deliver the asset and collect £9.75,
9. use £$\left(10 - 0.5e^{-0.02\times 0.5}\right) \times e^{0.02\times 1} \approx 9.70$ to repay the loan,
10. pocket the difference £$9.75 - 9.70 = 0.05$.

2(i)(a)
Such a portfolio is given by: 1 call option with strike 10,
-1 call options with strike 20.
1 call option with strike 30.
-2 call options with strike 40.
1 call option with strike 60.

2(i)(b)
The payoff function of the given put option is at least as big as the payoff function of the portfolio in part (a),
so the value of the put option is greater or equal to the value of the given portfolio at any time prior to expiration
Hence at the present we must have

$$p_{60} \geq c_{10} - c_{20} + c_{30} - 2c_{40} + c_{60}$$

2(ii)
Consider the following two portfolios:

- Portfolio A: one European put option plus one share.
- Portfolio B: an amount of cash equal to Xe^{-rT} deposited for T years at an interest rate of r.

After T years portfolio B will be worth X.

If, after T years, $S_T < X$, then the put option in portfolio A will be exercised; the share sold for X and the portfolio will be worth X.
Otherwise, if , after T years, $S_T \geq X$, the option is not exercised and the portfolio will be worth S_T.
So after T years portfolio A is worth $\max(S_T, X)$,
and the initial value of portfolio A must be no less that the initial value of portfolio B, which is just Xe^{-rT}.
But since sometimes portfolio A is worth more than portfolio B we have a strict inequality

$$p + S > Xe^{-rT}.$$

2(iii)(a) Consider a portfolio with x shares and short one call option.
The values of this portfolio in one year are $12x - 2$ or $4x$, and if we set these to be equal we get $x = 1/4$. So a portfolio with one share and short 4 call options will have the same value in one year in all states of the world.

2(iii)(b) We use $u = 1.5$, $d = 0.5$ and

$$q = \frac{e^{r\Delta t} - d}{u - d} \approx 0.53045$$

to construct the following two step binomial tree.

18

0

12

1.367

8

6

2.982

3

4

5

2

7

At each node the upper number is the stock price and the lower number is the option price.

2(iii)(b) A rational investor will exercise her option after one year if stock price goes down to 4.

3(i) The risk neutral valuation principle states that in valuing a derivative one may assume that:
(a) the value of a derivative producing a payoff at some time in the future is the present value of the expected value of the payoff
and
(b) the underlying asset has an expected return equal to the risk-free interest rate.

3(i)(a) A *Brownian motion* is a family of random variables

$$\{B_t \,|\, t \geq 0\}$$

on some probability space $(\Omega, \mathcal{F}, P)$
such that:
(a) $B_0 = 0$,
(b) for $0 \leq s < t$ the increment $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$,
(c) for any $0 \leq t_1 < t_2 < \cdots < t_n$ the increments

$$B_{t_1} - B_0, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_n - 1}$$

are independent random variables,
and
(d) For any $\omega \in \Omega$ the function $t \mapsto B_t(\omega)$ is continuous.

3(ii)(b) Let X_t be a stochastic process given by

$$dX = a(X,t)dt + b(X,t)dB.$$

Assume that $G(X,t)$ is twice continuously differentiable with respect to X and continuously differentiable with respect to t. The process $Y = G(X,t)$ is given by

$$dY = \left(\frac{\partial G}{\partial X}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2\right)dt + \frac{\partial G}{\partial X}b\; dB.$$

3(ii)(c) We have $F(t) = S(t)e^{r(T-t)}$, where $S(t)$ is the spot price of the stock at time t. If we regard F as a function of S and t, i.e., $F = F(S,t) = Se^{r(T-t)}$
we have

$$\frac{\partial F}{\partial S} = e^{r(T-t)}, \quad \frac{\partial^2 F}{\partial S^2} = 0 \quad \text{and} \quad \frac{\partial F}{\partial t} = -rSe^{r(T-t)}.$$

We assume that

$$dS = \mu S dt + \sigma S dB$$

so Ito's Lemma implies that

$$dF = \left(e^{r(T-t)}\mu S - rSe^{r(T-t)}\right)dt + e^{r(T-t)}\sigma S dB = (\mu - r)F dt + \sigma F dB,$$

i.e., F follows a geometric Brownian motion with drift $\mu - r$.

3(iii)(a)

$$\frac{\partial f}{\partial t} = rXe^{-r(T-t)}$$

$$\frac{\partial f}{\partial S} = -1$$

$$\frac{\partial^2 f}{\partial S^2} = 0$$

and

$$\begin{aligned}\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} &= rXe^{-r(T-t)} - rS \\ &= rf.\end{aligned}$$

3(iii)(b) Since $c(S,t)$ and $p(S,t)$ are prices of European options, the satisfy the Black-Scholes PDE. Since the Black-Scholes PDE is linear, and since $f(S,t)$ is a solution of it, the principle of superposition implies that the linear combination $g(S,t) = f(S,t) - c(S,t) + p(S,t)$ is also a solution of the Black-Scholes PDE.

3(iii)(c) Since $g(S,t)$ is a solution of the Black-Scholes PDE, $g(S,t)$ is the price at time t of a derivative on S which produces a payoff of $g(S_T, T)$ at time $T > t$.

Now

$$\begin{aligned} g(S_T, T) &= X - S_T + c(S_T, T) - p(S_T, T) \\ &= X - S_T + \max\{S_T - X, 0\} - \max\{X - S_T, 0\} \\ &= 0 \end{aligned}$$

So $g(S,t)$ is the price at time t of a derivative which pays *nothing* at time T. Since this derivative is worth the same as £0 at time T, its price at time $0 \leq t \leq T$ would be the present value at time t of £0 paid at time T, i.e., 0. We deduce that $g(S,t) = 0$ and so

$$c(S,t) + Xe^{-r(T-t)} = p(S,t) + S.$$

4(i)

(a)

The feasible set is the set of points in the σ-r plane with coordinates (σ, r) for which there exists an investment whose expected return is r and standard deviation of returns is σ.

(b)

The efficient frontier is the subset of the feasible set F consisting of all points (σ, r) for which for any other feasible portfolio $(\sigma', r') \in F$ if $\sigma = \sigma'$ then $r \geq r'$ and if if $r = r'$ then $\sigma \leq \sigma'$.

(c)

A market portfolio is a portfolio consisting entirely of risky investments which is efficient.

(Students may also define the market portfolio as the portfolio consisting of all possible investments, each with weight equal to the ratio between the total market value of the investment and the total value of all investments.)

(d)

The beta coefficient of an investment is C/σ_M^2 where C is the covariance between the returns of the given investment and the returns of the market portfolio and σ_M is the standard deviation of returns of the market portfolio.

(e)

The security market line is a line in the β-r plane consisting of all points (β, r) for all investments with beta-coefficient β and expected return r.

4(ii)(a)

The total value of the market is

$$1,000 \times 20 + 2,000 \times 10 = £40,000.$$

The relative value of Greed plc is

$$\frac{1,000 \times 20}{40,000} = \frac{1}{2}$$

and the relative value of Safety First Inc. is then

$$\frac{1}{2}.$$

The market portfolio consists of $£\frac{1}{2}$ invested in Greed plc shares and $£\frac{1}{2}$ invested in Safety First Inc.

4(ii)(b)

$$r_M = \frac{1}{2} \times 0.20 + \frac{1}{2} \times 0.05 = 12.5\%$$

The covariance C between the returns of the two investments is given by

$$C = 0.25 \times 0.5 \times 0.1 = 0.0125$$

and so

$$\sigma_M^2 = \left(\frac{1}{2} \times 0.5\right)^2 + \left(\frac{1}{2} \times 0.1\right)^2 + 2 \times \frac{1}{2} \times \frac{1}{2} \times 0.0125 \approx 0.07125$$

and

$$\sigma_M \approx 0.2667.$$

4(ii)(c) Let G, S and M be the random variables representing the returns of Greed Plc, Safety First Inc. and the market portfolio, respectively. The beta coefficient of Greed Plc is given by

$$\beta = \frac{\text{Covar}(G, M)}{\sigma_M^2} = \frac{\frac{1}{2}\,\text{Covar}(G, G) + \frac{1}{2}\,\text{Covar}(G, S)}{\sigma_M^2}.$$

We already computed $\text{Covar}(G, S) = 0.0125$ and so

$$\beta \approx \frac{\frac{1}{2} \times 0.5^2 + \frac{1}{2} \times 0.0125}{0.07125} \approx 1.8421.$$

4(ii)(d) Let r be the expected return of Greed Plc and denote the risk-free return with r_B. From the security market line we deduce that

$$r = r_B + (r_M - r_B)\beta \Rightarrow r_B = \frac{r - r_M\beta}{1-\beta} = .$$

$$\frac{0.2 - 0.125 \times 1.8421}{1 - 1.8421} \approx 3.59\%.$$

4(ii)(e) The desired portfolio must be efficient and hence lie on the capital market line. It will consist of £α deposited in a risk-free deposit account and of £$1-\alpha$ invested in the market portfolio, for some α. We solve $\alpha r_B + (1-\alpha)r_M = 0.08$, i.e., $\alpha \times 0.0359 + (1-\alpha) \times 0.125 = 0.08$ to obtain $\alpha \approx 50.50\%$.