MAS362 – 2013-14 Exam Solutions

1(i) The price S of the bond is the sum of the present value of all its future payments, and this equals

$$\sum_{i=0}^{\infty} 5e^{-0.03(i+1/2)} =$$

$$5e^{-0.03/2} \sum_{i=0}^{\infty} \left(e^{-0.03}\right)^i =$$

$$5e^{-0.03/2} \frac{1}{1-e^{-0.03}} \approx 166.66$$

1(ii) The present value I of the payments originating from the perpetual bond during the next N years is

$$\sum_{i=0}^{N-1} 5e^{-0.03(i+1/2)}$$

hence the N-year forward price is

$$F = e^{0.03N}(S-I) =$$

$$e^{0.03N}\left(\sum_{i=0}^{\infty} 5e^{-0.03(i+1/2)} - \sum_{i=0}^{N-1} 5e^{-0.03(i+1/2)}\right) =$$

$$e^{0.03N} \sum_{i=N}^{\infty} 5e^{-0.03(i+1/2)} =$$

$$\sum_{i=N}^{\infty} 5e^{-0.03(i+1/2-N)} =$$

$$\sum_{i=0}^{\infty} 5e^{-0.03(i+1/2)}$$

1(iii)

1. Short-sell the bond for £166.66.
2. Deposit £$5e^{-0.03\times 1/2} \approx 4.93$ for six months.
3. Deposit £$5e^{-0.03\times 3/2} \approx 4.78$ for eighteen months.
4. Deposit $166.66 - 4.93 - 4.78 = 156.95$ for two years.
5. At time $t = 1/2$ receive balance of £5 from first deposit and use it to pay first coupon.
6. At time $t = 3/2$ receive balance of £5 from second deposit and use it to pay second coupon.
7. At time $t = 2$ receive balance of £$156.95e^{0.03\times 2} \approx 166.66$, use forward agreement to buy a bond for £160, use the bond to close your short-position from step (1), pocket £6.66.

2(i)(a) (Similar to homework problem)
1 put option with strike 60. short 1 put option with strike 40. short 2 put options with strike 30. 2 put options with strike 20.

2(i)(b) (Similar to homework problem) The payoff function of the given call option is at least as big as the payoff function of the portfolio in part (a), so the value of the call option is greater or equal to the value of the given portfolio at any time prior to expiration Hence at the present we must have

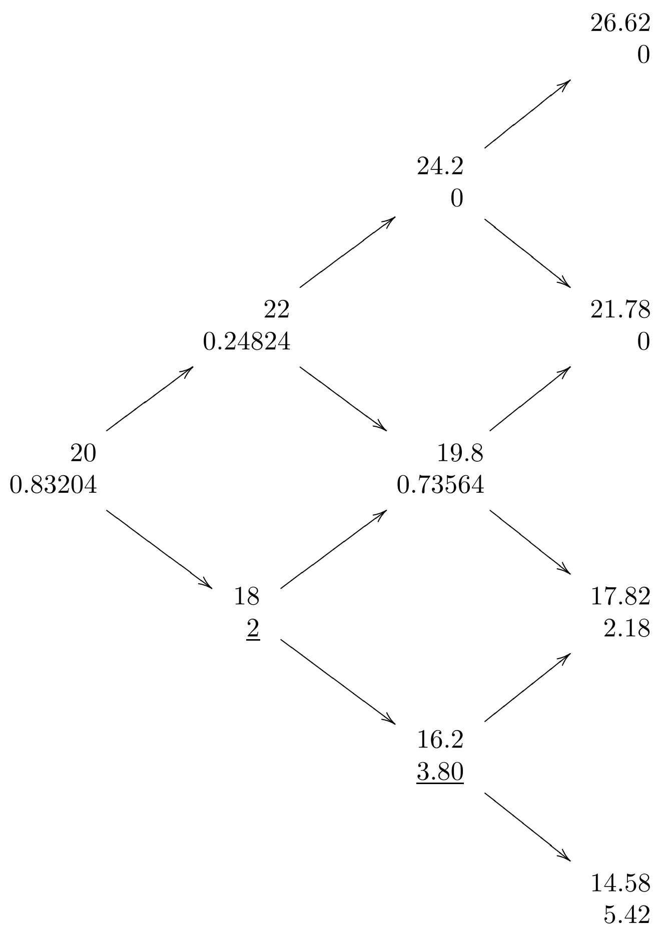
$$c_{10} \geq 2p_{20} - 2p_{30} - p_{40} + p_{60}$$

2(ii) (Bookwork) An American put option is a financial asset which gives its owner the right to sell a given asset at certain price by a certain date.

2(iii)(a) (Similar to homework problem) We use $u = 1.1$, $d = 0.9$ and

$$q = \frac{e^{r\Delta t} - d}{u - d} \approx 0.65227$$

to construct the following three step binomial tree.



At each node the upper number is the stock price and the lower number is the option price.
(+1) for each correct node.

2(iii)(b) (Similar to homework problem) A rational investor will exercise her option early only after one year if share price has decreased.

3(i)(a) *(Bookwork)*

Let S be the spot price of a certain stock at time t and let $G = G(s,t) = \log s$. Since

$$\frac{\partial G}{\partial s} = \frac{1}{s},$$
$$\frac{\partial^2 G}{\partial s^2} = \frac{-1}{s^2}$$
$$\frac{\partial G}{\partial t} = 0$$

and since

$$dS = \mu S dt + \sigma S dB$$

Ito's Lemma implies that

$$dG = \left(\frac{1}{S}\mu S - \frac{\sigma^2 S^2}{2S^2}\right) dt + \frac{\sigma S}{S} dB = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dB.$$

(Bookwork)

3(i)(b) At time t in a risk neutral world $\log S_T$ is normally distributed with mean

$$\log S_t + \left(r - \frac{\sigma^2}{2}\right)(T-t)$$

and variance $\sigma^2(T-t)$.

The event $S_T \leq a$ is equivalent to the event $\log S_T \leq \log a$ and so its probability is given by

$$\Phi\left(\frac{\log a - (\log S_t + \left(r - \frac{\sigma^2}{2}\right)(T-t))}{\sigma\sqrt{T-t}}\right) =$$

$$\Phi\left(\frac{\log\left(S_t/a\right) + \left(r - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

3(i)(c) (Homework problem)

The event $S_T \geq a$ is equivalent to the event $\log S_T \geq \log a$ and so its probability is given by

$$1 - \Phi\left(\frac{\log a - (\log S_t + \left(r - \frac{\sigma^2}{2}\right)(T-t))}{\sigma\sqrt{T-t}}\right) =$$

$$\Phi\left(\frac{\log\left(a/S_t\right) + \left(r - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

The expected value of the payoff of the derivative in a risk neutral world is $\Phi\left(\frac{\log(S_t/a) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)$ and its present value at time t is

$$e^{-r(T-t)}\Phi\left(\frac{\log(S_t/a) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

which is the value of the derivative by the risk-neutral valuation principle.

3(ii)(a) (Unseen)

$$\frac{\partial f}{\partial S} = 3e^{(2r+3\sigma^2)(T-t)}S^2, \quad \frac{\partial^2 f}{\partial S^2} = 6e^{(2r+3\sigma^2)(T-t)}S \quad \text{and} \quad \frac{\partial f}{\partial t} = -(2r+3\sigma^2)e^{(2r+3\sigma^2)(T-t)}S^3.$$

and

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = e^{(2r+3\sigma^2)(T-t)}S^3\left(-(2r+3\sigma^2) + 3r + 3\sigma^2\right) = rf.$$

3(ii)(a) (Unseen) Let $v(S,t)$ be the value of the derivative. Note that both f and v satisfies the Black-Scholes PDE, $v(0,t) = f(0,t) = 0$ for $0 \leq t \leq T$, and $v(S_T,T) = f(S_T,T) = S_T^3$. The uniqueness of solutions of the Black-Scholes PDE with the boundary conditions above implies that $v = f$.

4(i)(a)
A market portfolio is a portfolio consisting entirely of risky investments which is efficient. [Also, the portfolio consisting of all possible investments, each with weight equal to the ratio between the total market value of the investment and the total value of all investments.]
4(i)(b)
The Capital Market Line is the line on the σ-r plane consisting of all efficient investments. [Also the line connecting the points corresponding to the risk-free investment and the Market portfolio.]
4(i)(c)
The beta coefficient of an investment is C/σ_M^2 where C is the covariance between the returns of the given investment and the returns of the market portfolio and σ_M is the standard deviation of returns of the market portfolio.
4(ii)(a)
The slope of the capital market line is $(r_M - r_B)/\sigma_M$.
4(ii)(b)
Any feasible portfolio corresponding to the point (σ_P, r_P) on the σ-r plane cannot be above the capital market line, hence $(r_P - r_B)/\sigma_P \leq (r_M - r_B)/\sigma_M$. If two such portfolios existed, the concavity of the efficient frontier for risky investments would imply the existence of an efficient portfolio above the capital market lie, which is impossible.
4(ii)(c)
For any $0 \leq t \leq 1$, let portfolio Π_t consist of an investment of t in A and an investment of $1-t$ in M. c is given by $t \mapsto$ (std dev of returns of Π_t, expected return of Π_t), i.e.

$$t \mapsto \left(\sqrt{t^2\sigma_A^2 + 2t(1-t)\operatorname{Covar}(A,M) + (1-t)^2\sigma_M^2}, tr_A + (1-t)r_M\right) \quad (0 \leq t \leq 1).$$

4(ii)(d)
If c were not tangent to the market line, it would have to cross it, resulting in investments which are strictly preferable to an investment in the capital market line.
4(ii)(e)
Evaluate the derivative with respect to t

$$\left(\frac{2t\sigma_A^2 + (2-4t)\operatorname{Covar}(A,M) - 2(1-t)\sigma_M^2}{2\sqrt{t^2\sigma_A^2 + 2t(1-t)\operatorname{Covar}(A,M) + (1-t)^2\sigma_M^2}}, r_A - r_M\right)$$

at $t = 0$ to obtain the slope at point M: $\frac{r_A - r_M}{\operatorname{Covar}(A,M) - \sigma_M^2}\sigma_M$. But this slope must be equal to the slope of the capital market line, i.e., $\dfrac{r_A - r_M}{\operatorname{Covar}(A,M) - \sigma_M^2}\sigma_M = \dfrac{r_M - r_B}{\sigma_M}$ and we can rearrange this to obtain $r_A - r_B = \dfrac{r_M - r_B}{\sigma_M^2}\operatorname{Covar}(A,M) = \beta(r_M - r_B)$.