MAS362 – 2014-15 Exam Solutions

1(i) (a) $Y(0.5) = -\log(101.49/102)/0.5 \approx 1\%$.

(b) To find $Y(1)$ we solve

$$4e^{-Y(0.5)\times 0.5} + 104e^{-Y(1)\times 1} = 105.92$$

for the unknown $Y(1)$.

We have $Y(1) \approx -\log\big((105.92 - 4e^{-0.01\times 0.5})/104\big) \approx 2\%$.

To find $Y(1.5)$ we solve

$$3e^{-Y(0.5)\times 0.5} + 3e^{-Y(1)\times 1} + 103e^{-Y(1.5)\times 1.5} = 105.13$$

for the unknown $Y(1.5)$. We have

$$Y(1.5) \approx -\frac{1}{1.5}\log\left(\frac{105.15 - 3e^{-0.01\times 0.5} - 3e^{-0.02\times 1}}{103}\right) \approx 2.5\%.$$

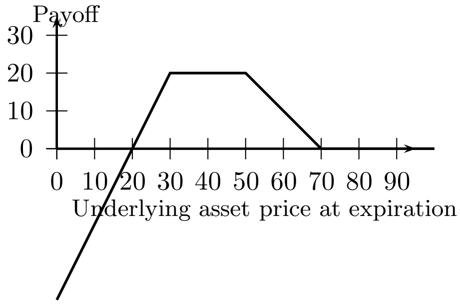
(c) The present value of the income generated by the bond during the life of the forward contract is $I = 3e^{-0.01/2} + 3e^{-0.02} \approx 5.9256$ and so the forward price is given by

$$(105.13 - I)e^{Y(1)\times 1} \approx 101.21.$$

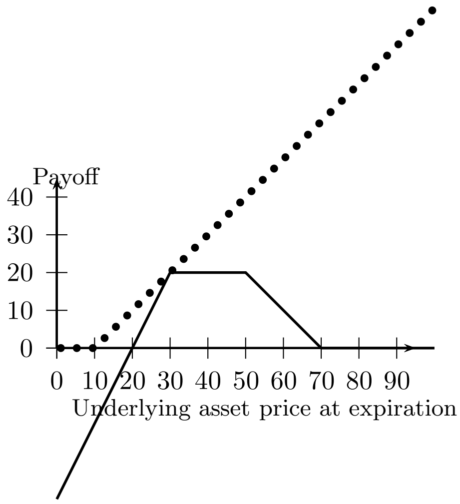
(d)

1. Enter agreement with a short position,
2. borrow £$3e^{-0.01/2}$ for 6 months, £$3e^{-0.02}$ for one year and £$105.13 - 3e^{-0.01/2} - 3e^{-0.02}$ for one year,
3. buy the 1.5 year bond for £105.13,
4. and wait.
5. After 6 months, collect £3 coupon, and repay first loan.
6. After another 6 months, collect £3 coupon, and repay second loan;
7. deliver the bond for £104 and repay balance of third loan amounting to $(105.13 - 3e^{-0.01/2} - 3e^{-0.02})e^{0.02\times 1} \approx 101.21$;
8. Pocket £$104 - 101.21 > 0$.

2(i)(a)



2(i)(b)
If we add the graph of the payoff of the call option to the graph



we can observe that the payoff of the call option is at least as big as that of the portfolio, so the present price c_{10} of the call option is at least as big as the present price of the portfolio, which equals $-2p_{30} - p_{50} + p_{70}$.

2(ii)(a)
Consider a portfolio consisting of δ shares and one put option. The payoff of the option when stock price is 10 is $X - 10$ and zero otherwise, so the prices of the portfolio in one year is $X - 10 + 10\delta$ when the stock price is 10, and 20δ when stock price is 20. Setting these two values to be equal and solving for δ gives $\delta = (X - 10)/10$.

2(ii)(b)
If the put option in the portfolio is not exercised immediately, the portfolio will be worth $20\delta = 2(X - 10)$ is one year,
hence its present value is $2e^{-0.03}(X - 10)$.

If the option is rationally exercised immediately, $X > 15$ and we obtain a payoff of $X - 15$, and the portfolio will be worth $X - 15 + 15\delta$.

The option would be exercised immediately when $X > 15$ and $X - 15 + 15\delta > 2e^{-0.03}(X - 10)$.

We solve the inequality: $X - 15 + 15(X - 10)/10 > 2e^{-0.03}(X - 10)$
which simplifies to $X(1 + 15/10 - 2e^{-0.03}) > 30 - 20e^{-0.03}$ and $X > 18.94$.

We conclude that the option should be exercised if $X > 18.94$.

3(i)(a) A *Brownian motion* is a family of random variables

$$\{B_t \,|\, t \geq 0\}$$

on some probability space $(\Omega, \mathcal{F}, P)$
such that:
(a) $B_0 = 0$,
(b) for $0 \leq s < t$ the increment $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$,
(c) for any $0 \leq t_1 < t_2 < \cdots < t_n$ the increments

$$B_{t_1} - B_0, B_{t_2} - B_{t_1}, \ldots, B_{t_{n-1}} - B_{t_n}$$

are independent random variables,
and
(d) For any $\omega \in \Omega$ the function $t \mapsto B_t(\omega)$ is continuous.

3(ii)(b) Let X_t be a stochastic process given by

$$dX = a(X,t)dt + b(X,t)dB.$$

Assume that $G(x,t)$ is twice continuously differentiable with respect to x and continuously differentiable with respect to t. The process $Y = G(X,t)$ is given by

$$dY = \left(\frac{\partial G}{\partial X}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2\right)dt + \frac{\partial G}{\partial X}b\; dB.$$

3(ii)
(a) Apply Ito's Lemma with $a(S,t) = \mu S$ and $b(S,t) = \sigma S$:
the process followed by f is

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S dB.$$

(b) The discrete version of the equation in (a)

$$\Delta f \approx \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S \Delta B.$$

where ΔB is a normally distributed random variable with zero mean and variance Δt.
Now

$$\begin{aligned}
\Delta\Pi &= \frac{\partial f}{\partial S}\Delta S - \Delta f \\
&\approx \frac{\partial f}{\partial S}(\mu S\Delta t + \sigma S\Delta B) - \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t - \frac{\partial f}{\partial S}\sigma S\Delta B \\
&= \left(-\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 - \frac{\partial f}{\partial t}\right)\Delta t.
\end{aligned}$$

(c) Since $\Delta\Pi$ in deterministic for very small Δt, it must grow at the same rate as risk-free deposits, i.e.,

$$\Delta\Pi = e^{r\Delta t}\Pi - \Pi$$

and for very small $\Delta t > 0$ we obtain $\Delta\Pi \approx (1 + r\Delta t)\Pi - \Pi$ and $\Delta\Pi \approx r\Pi\Delta t$.
(d) Now as $\Delta t \to 0$ we replace Δt with an infinitesimal dt and we combine (b) and (c) to obtain

$$\left(-\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 - \frac{\partial f}{\partial t}\right)dt = r\left(\frac{\partial f}{\partial S}S - f\right)dt$$

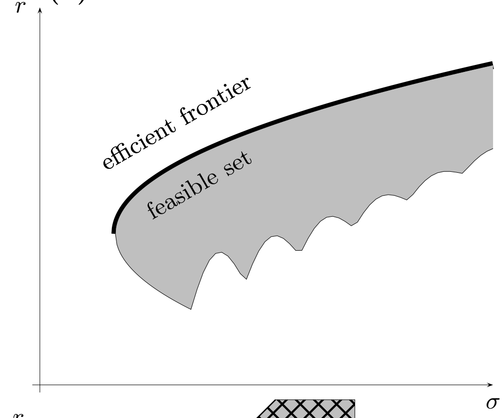
which we rewrite as

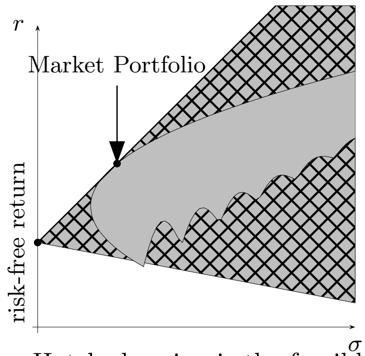
$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf.$$

(e) It doesn't.
The risk aversion of investors is expressed in the value of μ: higher risk aversion would demand higher expected returns μ for a given level of volatility σ.
Since μ does not occur in the Black Scholes PDE, risk aversion does not affect the value of the derivative.

4(i)(a)
A market portfolio is a portfolio consisting entirely of risky investments which is efficient. [Also, the portfolio consisting of all possible investments, each with weight equal to the ratio between the total market value of the investment and the total value of all investments.]
4(i)(b)
The Capital Market Line is the line on the σ-r plane consisting of all efficient investments. [Also the line connecting the points corresponding to the risk-free investment and the Market portfolio.]
4(i)(c)
The beta coefficient of an investment is C/σ_M^2 where C is the covariance between the returns of the given investment and the returns of the market portfolio and σ_M is the standard deviation of returns of the market portfolio.

4(ii)





Hatched region is the feasible set. **(+3)**

4(iv) The capital market line is given by

$$r - 0.03 = \frac{0.045 - 0.03}{0.37}\sigma \Rightarrow r = 0.04054\sigma + 0.03.$$

The beta coefficient of Stock 1 is

$$\beta_1 = \frac{\text{Covar}(\text{Stock 1}, M)}{\sigma_M^2} = \frac{\rho(\text{Stock 1}, M)\sigma_1}{\sigma_M} = \frac{0.7 \times 0.8}{0.37} \approx 1.5135.$$

We obtain the expected return using the security market line

$$r_1 = r_B + (r_M - r_B)\beta_1 = 0.03 + (0045 - 0.03) \times 1.5135 \approx 0.052703.$$

The beta coefficient β_2 of Stock 2 must satisfy

$$r_2 = r_B + (r_M - r_B)\beta_2 \Rightarrow \beta_2 = \frac{0.05 - 0.03}{0.045 - 0.03} \approx 1.3333$$

and so the correlation with the market portfolio is

$$\rho(\text{Stock 2}, M) = \frac{\text{Covar}(\text{Stock 2}, M)}{\sigma_2 \sigma_M} = \frac{\beta_2 \sigma_M}{\sigma_2} = 1.3333 \times 0.37/0.63 \approx 0.78305.$$

The beta coefficient β_3 of Stock 3 must also satisfy

$$r_3 = r_B + (r_M - r_B)\beta_3 \Rightarrow \beta_3 = \frac{0.038 - 0.03}{0.045 - 0.03} = 0.53333$$

but

$$\beta_3 = \frac{\text{Covar}(\text{Stock } 3, M)}{\sigma_M^2} = \frac{\rho(\text{Stock } 3, M)\sigma_3}{\sigma_M}$$

so

$$\sigma_3 = \frac{\sigma_M \beta_3}{\rho(\text{Stock } 3, M)} = \frac{0.37 \times 0.53333}{0.9} = 0.21926.$$