**MAS362 – 2015-16 Exam Solutions**

**(1)** **(i)** $Y(0.5) = -\log(99.005/100)/0.5 \approx 2\%$.
**(ii)** To find $Y(1)$ we solve

$$3e^{-Y(0.5)\times 0.5} + 103e^{-Y(1)\times 1} = 102.926$$

for the unknown $Y(1)$. We have $Y(1) \approx -\log\big((102.926 - 4e^{-0.02\times 0.5})/103\big) \approx 3\%$.
To find $Y(2)$ we solve

$$2e^{-Y(0.5)\times 0.5} + 2e^{-Y(1)\times 1} + 2e^{-Y(1.5)\times 1.5} + 102e^{-Y(2)\times 2} = 100.93$$

for the unknown $Y(2)$. We have

$$Y(2) \approx -\frac{1}{2}\log\left(\frac{100.93 - 2e^{-0.02\times 0.5} - 2e^{-0.03\times 1} - 2e^{-0.032\times 1.5}}{102}\right) \approx 3.5\%.$$

**(iii)** The forward rate for deposits from 1 to 2 years is $(2\times 3.5\% - 1\times 3\%)/(2-1) = 4\% > 3.75\%$, so we enter the forward agreement as a borrower.

1. We borrow £$1,000,000e^{-0.03}$ for 1 year and
2. deposit £$1,000,000e^{-0.03}$ for 2 years.
3. After 1 year we borrow £$1,000,000$ for 1 year at an interest rate of 3.75% and
4. use it to repay our loan.
5. After an additional year we obtain balance of the deposit which amounts to

   $$£1,000,000e^{-0.03}e^{0.035\times 2} \approx 1,040,811$$

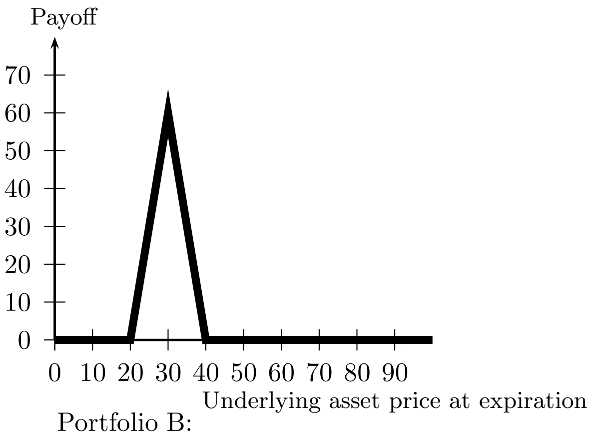
   and **(+1)**
6. repay the balance of our loan which amounts to £$1,000,000e^{0.0375\times 1} \approx 1,038,212$ and
7. we pocket the difference of £$1,040,811 - 1,038,212 = 2599 > 0$.

**(iv)** The present value income provided by the bond during the life of the froward contract is $I = 2e^{-Y(0.5)\times 0.5} + 2e^{-Y(1)\times 1} \approx 3.921$, and the forward price of the bond is $(100.93 - 3.921)e^{0.03\times 1} \approx 99.96$.

## (2) Variation on class examples and homework problems

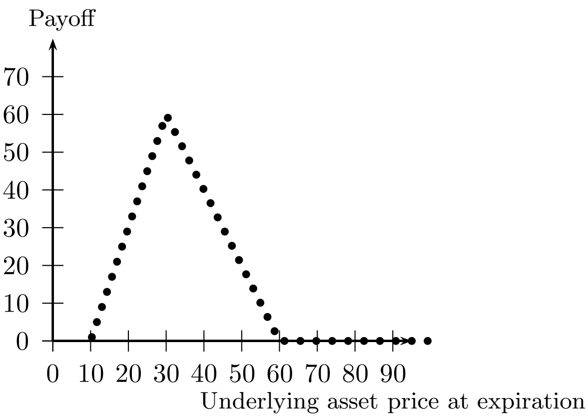
**2(i)(a)**

Portfolio A:



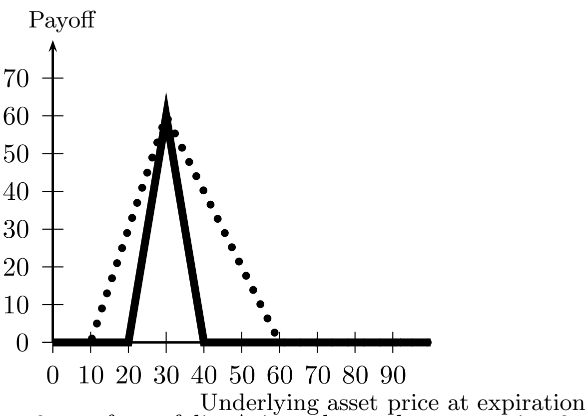

**(+3)**

Portfolio B:




**2(i)(b)**

The payoff of portfolio B at expiration is always no less than that of portfolio A:



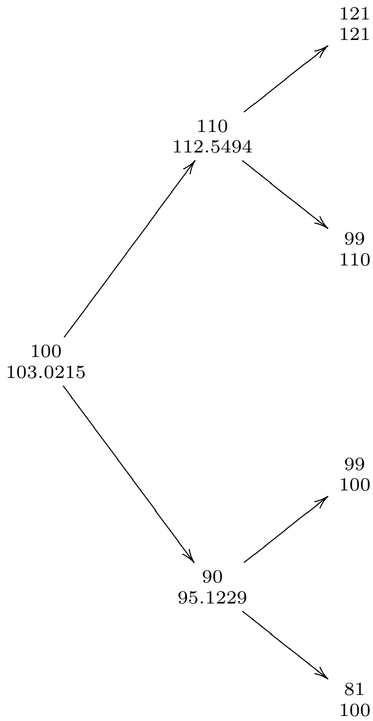

so the spot price $6p_{20} - 12p_{30} + 6p_{40}$ of portfolio A is at least the spot price $3c_{10} - 5c_{30} + 2c_{60}$ of portfolio B,

**2(ii)(a) (Unseen)**

The risk-neutral probability of stock price going up is

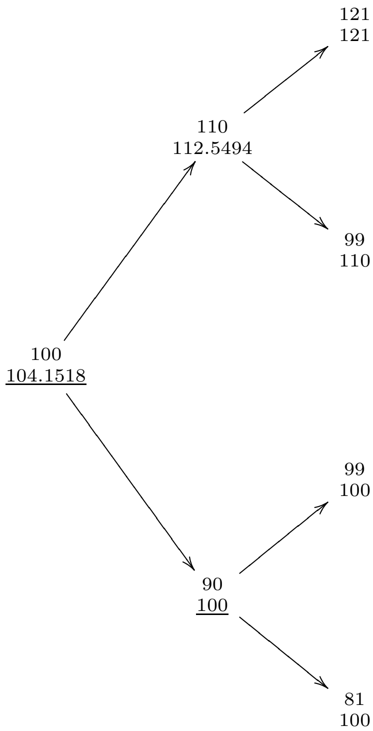
$$\frac{e^{0.05} - 0.9}{1.1 - 0.9} \approx 0.756355$$

and we obtain the following tree.




At each node the upper number is the stock price and the lower number is the option price.

**2(ii)(b) (Unseen)**



**3(i)**

The risk neutral valuation principle states that in valuing a derivative one may assume that: (a) the value of a derivative producing a single payoff at some time in the future which is a function of the price of the underlying asset equals the present value of the expected value of that payoff and (b) the underlying asset has an expected return equal to the risk-free interest rate.

**3(ii)** Let $X_t$ be a stochastic process given by

$$dX = a(X,t)dt + b(X,t)dB.$$

Assume that $G(x,t)$ is twice continuously differentiable with respect to $x$ and continuously differentiable with respect to $t$. The process $Y = G(X,t)$ is given by

$$dY = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\ dB.$$

**3(iii)(a)**

Let $S$ be the spot price of a certain stock at time $t$ and let $G = G(s,t) = \log s$. Since

$$\frac{\partial G}{\partial s} = \frac{1}{s},$$
$$\frac{\partial^2 G}{\partial s^2} = \frac{-1}{s^2}$$
$$\frac{\partial G}{\partial t} = 0$$

and since

$$dS = \mu S dt + \sigma S dB$$

Ito's Lemma implies that

$$dG = \left(\frac{1}{S}\mu S - \frac{\sigma^2 S^2}{2S^2}\right)dt + \frac{\sigma S}{S}dB = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dB.$$

*(Unseen: variation on bookwork)*

**3(iii)(b)** At time 0 in a risk neutral world $\log S_T$ is normally distributed with mean

$$\log S + \left(r - \frac{\sigma^2}{2}\right)T$$

and variance $\sigma^2 T$.

The event $a \le S_T \le b$ is equivalent to the event $\log a \le \log S_T \le \log b$ and so its probability is given by

$$\Phi\left(\frac{\log b - (\log S + \left(r - \frac{\sigma^2}{2}\right)T)}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{\log a - (\log S + \left(r - \frac{\sigma^2}{2}\right)T)}{\sigma\sqrt{T}}\right) =$$

$$\Phi\left(\frac{\log(b/S) - \left(r - \sigma^2/2\right)T)}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{\log(a/S) - \left(r - \sigma^2/2\right)T}{\sigma\sqrt{T}}\right).$$

**3(iii)(c)** The risk-neutral valuation principle shows that the price of the derivative is

$$e^{-rT}\left(\Phi\left(\frac{\log(b/S) - \left(r - \sigma^2/2\right)(T)}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{\log(a/S) - \left(r - \sigma^2/2\right)(T)}{\sigma\sqrt{T}}\right)\right).$$

**4(i)**

**(a)**
The market portfolio will consist of a £$t$ invested in A and £$1-t$ invested in B. Such a portfolio has expected returns $r_t = 0.25t + 0.1(1-t) = 0.15t + 0.1$ and standard deviation of returns $\sigma_t = \sqrt{0.5^2t^2 + 0.1^2(1-t)^2} = \sqrt{0.26t^2 - 0.02t + 0.01}$. We maximize the function $f(t) = \frac{r_t - 0.06}{\sigma_t}$; we compute

$$f'(t) = \frac{0.15\sqrt{0.26t^2 - 0.02t + 0.01} - \frac{(0.15t+0.04)}{2\sqrt{0.26t^2-0.02t+0.01}}}{0.26t^2 - 0.02t + 0.01} =$$

$$\frac{0.15(0.26t^2 - 0.02t + 0.01) - \frac{(0.15t+0.04)}{2}}{(0.26t^2 - 0.02t + 0.01)^{3/2}} = \frac{0.0018 - 0.0119t}{(0.26t^2 - 0.02t + 0.01)^{3/2}}$$

and solve $f'(t) = 0$ to obtain $t \approx 0.15966$.

**(b)**
Let $x$ denote the total value of B. We have $x/(10^8 + x) = 1 - 0.15966$. Solving for $x$ gives $x = 526,330,953$.

**4(ii)**

**(a)**
The slope of the capital market line is $(r_M - r_B)/\sigma_M$.

**(b)**
For any $0 \leq t \leq 1$, let portfolio $\Pi_t$ consist of an investment of $t$ in A and an investment of $1-t$ in M. $c$ is given by $t \mapsto$ (std dev of returns of $\Pi_t$, expected return of $\Pi_t$), i.e.

$$t \mapsto \left(\sqrt{t^2\sigma_A^2 + 2t(1-t)\operatorname{Covar}(A,M) + (1-t)^2\sigma_M^2}, tr_A + (1-t)r_M\right) \quad (0 \leq t \leq 1).$$

**(c)**
If $c$ were not tangent to the market line, it would have to cross it, resulting in investments which are strictly preferable to an investment in the capital market line.

**(d)**
Evaluate the derivative with respect to $t$

$$\left(\frac{2t\sigma_A^2 + (2-4t)\operatorname{Covar}(A,M) - 2(1-t)\sigma_M^2}{2\sqrt{t^2\sigma_A^2 + 2t(1-t)\operatorname{Covar}(A,M) + (1-t)^2\sigma_M^2}}, r_A - r_M\right)$$

at $t = 0$ to obtain the slope at point $M$: $\frac{r_A - r_M}{\operatorname{Covar}(A,M) - \sigma_M^2}\sigma_M$. But this slope must be equal to the slope of the capital market line, i.e., $\dfrac{r_A - r_M}{\operatorname{Covar}(A,M) - \sigma_M^2}\sigma_M = \dfrac{r_M - r_B}{\sigma_M}$ and we can rearrange this to obtain $r_A - r_B = \dfrac{r_M - r_B}{\sigma_M^2}\operatorname{Covar}(A,M) = \beta(r_M - r_B)$.