Interest, present value and bonds

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What is the price of £1000 paid in one year?

Time is Money

Money is a resource, just as oil and labour.
It can be invested in profit-making ventures.
People running such a venture need to raise capital, often by *borrowing* it.
Lenders forego the pleasure of spending their money now in exchange for a payment, i.e., *interest.*

Money+Time=Money+Interest

If you have cash available, you can lend it
(say, to a bank, by depositing it in a savings account.)

Your loan will then accrue interest.

Interest is usually compounded, i.e.,
interest is applied to the original amount deposited plus the interest gained thus far.

Interest

Interest rates apply for a given period, usually a year,
and interest is often compounded a fixed number of times in that period.

Convention: If an amount A is invested for n years at a yearly interest rate r
and, if the interest is compounded m times a year,
the terminal value of the deposit is $A\left(1+\frac{r}{m}\right)^{mn}$.
E.g., if I borrow £1000 from you for three years and I agree to pay 5% annual interest compounded every six months, I will repay you

$$£1000(1+\frac{0.05}{2})^6 \approx £1159.69.$$

Present Value

How much are people willing to pay now in order to receive £1 in 10 years?
Call that amount X.
Suppose 10-year deposits pay 3% per year compounded yearly.
If you deposit $\pounds X$ for 10 years you end up with
$\pounds X \times 1.03^{10} \approx 1.34 \times X$
If you deposit $\pounds 1/1.34 \approx 0.74$ now, you obtain £1 in 10 years.
This is the present value of £1 paid in 10 years.
If people pay $A > 1.03^{-10}$, take their money, deposit it, collect $A \times 1.03^{10} > 1$ after 10 years, pay £1, pocket profit $A \times 1.03^{10} - 1 > 0$.
If people pay $A < 1.03^{-10}$, borrow $\pounds A$, and exchange it for £1 in 10 years, wait 10 years, collect £1, repay $A \times 1.03^{10} < 1$ pocket profit $1 - A \times 1.03^{10} > 0$.

wait a minute...

A guaranteed profit was achieved with no investment whatsoever.

But **there are no free lunches!**

We assume in this course that arbitrage opportunities *do not exist.*

This is reasonable: if arbitrage exists, arbitrage trading will occur and prices will quickly change to eliminate these arbitrage opportunities.

So in the example, £1 paid in 10 years is worth 1.03^{-10} now,

Example: a mortgage

Consider a £100,000 mortgage paying a fixed annual interest rate of 6% which is compounded monthly. What are the monthly repayments if the mortgage is to be repaid in 20 years?

Call the monthly repayment P.

Question: How much is the bank willing to pay you now in order to receive £P in m months?

Answer: $P \times \left(1 + \frac{0.06}{12}\right)^{-m}$.

The present value of all monthly payments of £P for 20 years is

$\sum_{m=1}^{20\times 12} P \left(1 + \frac{0.06}{12}\right)^{-m}$ and we must have

$$100,000 = P \sum_{m=1}^{240} \left(1 + \frac{0.06}{12}\right)^{-m} = P \sum_{m=1}^{240} \left(\frac{200}{201}\right)^{m}$$

Solving for P gives

$$P = \frac{100,000}{\sum_{m=1}^{240} \left(\frac{200}{201}\right)^{m}} = \frac{100,000}{\frac{200}{201} \frac{(200/201)^{240} - 1}{-1/201}} = \frac{-100,000}{200\left(\left(\frac{200}{201}\right)^{240} - 1\right)}$$

≈ 716.43

Continuously compounded interest

Now suppose that we increase the frequency of the compounding of the interest;
the limiting terminal value will converge to

$$\lim_{m\to\infty} A\left(1+\frac{r}{m}\right)^{mn} = A\left(\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{m}\right)^{n} = Ae^{rn}.$$

Thus if we borrow £A for t-years accruing a yearly interest of r compounded continuously, we pay back £Ae^{rt}.

We want to study situations where interest accrues for short or irregular periods of time. It is easier then to use continuously compounded interest rates.

Unless stated otherwise, we always assume that interest is compounded continuously.

Note: the same payment of interest can be described using different interest-compounding conventions. For example, a deposit of £100 yielding a balance of £110 in one year pays

- 10% annual interest compounded yearly,
- $2(\sqrt{1.1} - 1)$ annual interest compounded every six months,
- $\log 1.1$ annual interest compounded continuously.

Bonds

Governments and corporations raise capital by issuing bonds.
Bonds are contracts in which the issuer commits to pay the holder of the bond payments of certain amounts on certain dates.

A bond has:
a face value which is the amount of the final payment,
a maturity date which is the date when final payment is made,
and coupons which are a collection of amounts of payments and their dates.

Some UK gilts.
Visit the *Gilts in issue page* of the UK Debt Management Office

Example:

A bond with face value of £100, paying 6% annual interest, with 2 years to maturity and with semiannual coupons entitles its owner to the following payments:
£3 in 6 months,
£3 in 12 months,
£3 in 18 months,
and £103 in 24 months.

Zero coupon bonds

A zero coupon bond has no coupons.
Not common, but useful for theoretical purposes.
The price P of a £1 face value zero coupon bond with maturity in t years
is precisely the amount of money you are willing to pay now in order to be receive £1 in t years.
I.e., P is the present value of £1 paid t years into the future.

Interest rates and bond prices

There is a close link between prices of bonds and interest rates paid on cash deposits:
assume that one can obtain borrow and lend for t years at an annual interest of r.
Then the price P of a £1 face value zero coupon bond with maturity in t years cannot exceed e^{-rt},
otherwise issue such a bond, sell it for £P, and deposit this amount for t years.
After t years obtain a terminal value of $Pe^{rt} > 1$, pay the face value of £1 to the owner of the bond, and pocket $Pe^{rt} - 1 > 0$!

wait a minute...

A guaranteed profit was achieved with no investment whatsoever.

But **there are no free lunches!**

$P = e^{-rt}$

If $P < e^{-rt}$ do the opposite:
borrow £P for t years and buy the bond; wait for t years, return the terminal value of the loan, $Pe^{rt} < 1$, receive the face value of £1, pocket $1 - Pe^{rt} > 0$.

A free lunch!

So we have just proved that

$$P = e^{-rt}.$$

(both $P > e^{-rt}$ and $P < e^{-rt}$ lead to arbitrage opportunities, hence do not occur.)

Discount curves and yield curves

Some definitions and jargon:
A discount curve is a function of time $P(t)$ whose value at any $t \geq 0$ is the present value of one unit of currency paid in t years.
Equivalently, $P(t)$ is the price of a zero coupon bond with face value 1 maturing in t years.
A yield curve is a function $Y(t)$ whose value at any $t > 0$ is the interest rate paid on t-year deposits.
The values of $Y(t)$ are also called spot (interest) rates or yields.
(A yield curve)

$P(t) = e^{-Y(t)t}$

We proved the following:
For any $t > 0$ for which there is a zero coupon bond with maturity in t years and a t-year deposit account
we have $P(t) = e^{-Y(t)t}$.
We shall henceforth assume that for any $t > 0$ there is a zero coupon bond with maturity in t years and a t-year deposit account.

The bootstrap method

We now describe a method of producing yield curves based on the prices of bonds.

Consider the following bonds with semi-annual coupons:

Face value (£)	Maturity (years)	Annual interest	Price
100	0.25	0	98.3
100	0.5	0	96.5
100	1	0	93.7
100	1.5	4	95.5
100	2	6	97.2

We can translate the first three lines in the table to

$P(0.25) = 0.983$, $Y(0.25) = -\frac{\log 0.983}{0.25} \approx 6.86\%$

$P(0.5) = 0.965$, $Y(0.5) \approx 7.13\%$

$P(1) = 0.937$, $Y(1) \approx 6.51\%$.

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100	2	6	97.2

The fourth line implies that $2P(0.5) + 2P(1) + 102P(1.5) = 95.5$ and solving for $P(1.5)$ we obtain $P(1.5) \approx 0.89898$ and $Y(1.5) \approx 7.10\%$.

The fifth line implies that $3P(0.5) + 3P(1) + 3P(1.5) + 103P(2) = 97.2$ and solving for $P(2)$ we obtain $P(2) \approx .8621073672$ and $Y(2) \approx 7.42\%$.

Risk

Banks go bankrupt (e.g., Iceland bank crisis) and governments default on their debt (e.g., Russian debt default), The price of bonds and the rates of deposit reflect this:
risky bonds are *cheaper* and deposits in shaky banks pay *higher* interest.
The same is true of secured loans, e.g., mortgages, compared to non-secured loans.

Example: Euro zone bond prices

Consider the following 10-year Euro-zone government bond yields) (21/9/2010) :

Government	10-year yield
Germany	2.45%
Denmark	2.53%
Netherlands	2.65%
Finland	2.72%
France	2.80%
Austria	2.92%
Belgium	3.29%
Italy	3.91%
Spain	4.21%
Portugal	6.33%
Greece	11.44%

Without risk of default these yields would have to be equal!

Bond yields are giving us information about the perception of relative risks of defaults of these governments: Germany is rated the least risky while Greece is the most risky.

Risk-free interest rates

When working with interest rates we will disregard risk.
We always assume that for every currency there exists a risk-free institution issuing bonds in that currency.
Our yield curves and discount factors will refer to these bonds.
We also assume that everyone can both deposit and borrow any amount of cash for any maturity date at these risk-free rates.

10 minute Quiz! (2002-3 exam)

Consider the following five bonds with face value of £100:

Time to maturity (in years)	Annual interest (paid every 6 months)	Bond price (in £)
0.25	0	99.
0.5	0	97.8
1.	0	95.5
1.5	8%	104.5
2.	12%	113.

(a) Find the 0.25, 0.5 and 1-year spot interest rates. (3 marks)

(b) Use the bootstrap method to find the 1.5 and 2-year spot interest rates. (8 marks)

Forward rates

You plan to deposit £1 t_1 years from now and to withdraw the balance t_2 years from now ($t_2 > t_1$),
but you would like to agree now on the interest rate r_{12} for this deposit.

What should r_{12} be?

Let the spot interest years for t_1 and t_2 year deposits be r_1 and r_2. You could borrow £$e^{-r_1 t_1}$ for t_1 years and deposit this amount for t_2 years. At time $t = t_1$ you repay the balance of your loan, which will be $e^{-r_1 t_1} e^{r_1 t_1} = 1$ and at time $t = t_2$ you will receive $e^{-r_1 t_1} e^{r_2 t_2} = e^{r_2 t_2 - r_1 t_1}$.

This strategy is equivalent to depositing £1 at time t_1 until time t_2.
So we should expect to receive

$$e^{r_{12}(t_2-t_1)} = e^{r_2 t_2 - r_1 t_1} \Rightarrow r_{12} = \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1}.$$

We shall refer to r_{12} as the
forward rate for the period from t_1 to t_2.

Forward rate agreements

A forward rate agreement is a contract in which one party agrees to pay the other party a pre-specified interest rate on a deposit in a future period of time.

Proposition 1.1: *In a market with no arbitrage opportunities the interest rates of forward rate agreements are equal to the corresponding forward rates.*

Proof: (Another no-arbitrage argument!) Let the forward rate agreement time period start in t_1 years and end in t_2 years, let r be the pre-specified interest rate and let

$$r_{12} = \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1}$$

be the corresponding forward rate for the period from t_1 to t_2.

If $r > r_{12}$ enter the agreement as a depositor, borrow $e^{-r_1 t_1}$ for t_2 years and deposit this amount for t_1 years.
At time $t = t_1$ receive the balance of your deposit which will be $e^{-r_1 t_1} e^{r_1 t_1} = 1$, and deposit it until $t = t_2$ earning an interest rate of r.
At time $t = t_2$ obtain the balance of your deposit which will equal $e^{r(t_2 - t_1)}$ and repay the balance of your loan, which will be $e^{-r_1 t_1} e^{r_2 t_2} = e^{r_2 t_2 - r_1 t_1}$.
Since

$$r > \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1} \Rightarrow e^{r(t_2 - t_1)} > e^{r_2 t_2 - r_1 t_1}$$

you pocket the difference $e^{r(t_2 - t_1)} - e^{r_2 t_2 - r_1 t_1} > 0$.

If $r < r_{12}$ adopt the opposite strategy: enter the agreement as a borrower, borrow $e^{-r_1 t_1}$ for t_1 years and deposit this amount for t_2 years.
At time $t = t_1$ borrow £1 until $t = t_2$ paying an interest rate of r, and use this £1 to repay your loan.
At time $t = t_2$ receive the balance of your deposit which will be $e^{-r_1 t_1} e^{r_2 t_2} = e^{r_2 t_2 - r_1 t_1}$, and use it to repay the balance of your second loan which will be $e^{r(t_2 - t_1)}$.
Now

$$r < \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1} \Rightarrow e^{r(t_2 - t_1)} < e^{r_2 t_2 - r_1 t_1}$$

and you pocket the difference $e^{r_2 t_2 - r_1 t_1} - e^{r(t_2 - t_1)} > 0$.

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Suppose that you are offered by a risk free institution the opportunity to deposit or borrow £1000 in six months for a period of six months earning an interest rate of 5%. Describe in detail an arbitrage opportunity available to you. (11 marks)

The End