Options

Moty Katzman

September 19, 2014

What are options?

Options are contracts conferring certain rights regarding the buying or selling of assets.

A European call option gives the owner the right to *buy* its underlying asset at a certain price *on* a certain date.

A European put option gives the owner the right to *sell* its underlying asset at a certain price *on* a certain date.

The date on which the owner of European options can exercise his right is called the expiration date and the price is called the strike price.

What are options?

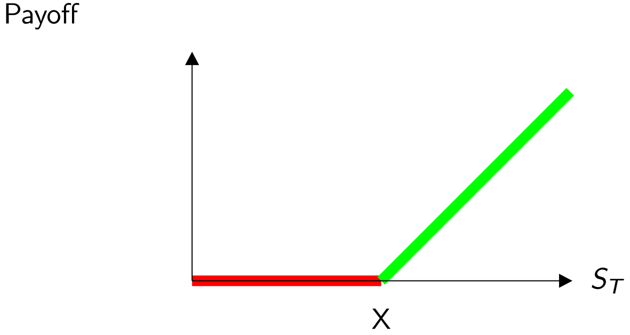
An American call option gives the owner the right to buy its underlying asset at a certain price *by* a certain date.

An American put option gives the owner the right to sell its underlying asset at a certain price *by* a certain date.

The date by which the owner of American options can exercise his right is called the expiration date and the price is called the strike price.

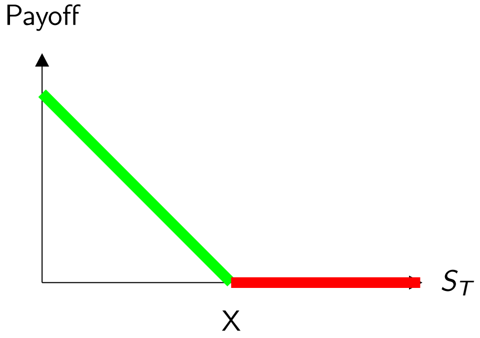
(Options on the NYSE)

The payoffs of options



Consider a European call option with strike price X which has just expired at time $t = T$ and let S_T be the spot price of the underlying asset at the expiration of the option.
If $S_T > X$ the holder of the option will buy the asset for X and sell it for S_T, thus getting a payoff of $S_T - X$ from the option.
If $S_T \leq X$ the option holder will not exercise it; in this case the option generates no payoff.

Similarly, the payoff of a corresponding European put option is described by the following graph:

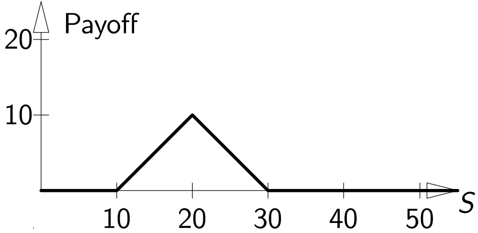


Example: Strangles

A strangle is a portfolio consisting of two options with same underlying asset and same expiration date.
One option is a put with strike price X_1 and the other is a call option with strike price $X_2 > X_1$.
Sketch the payoff function of this portfolio.
Why would someone hold this portfolio?

10 minute Quiz! (2003-4 Exam)

Describe a portfolio consisting entirely of European style put options on a stock with different strike prices and same expiration time $T > 0$, and whose payoff function at time T as a function of S, the spot price of the stock at time T, is described in the graph below. (6 marks)



Let p_{10}, p_{20} and p_{30} be the prices of the above put options with strike prices $10, 20$ and 30, respectively. Show that
$p_{20} \leq \frac{1}{2}(p_{10} + p_{30})$. (4 marks)

Elementary inequalities satisfied by option prices

We consider different option types on the same stock with spot price S, expiring in T years and with strike price X.
We will denote with c and p the current prices of European call and put options,
and with C and P the current prices of American call and put options.

Proposition 3.1:
(1) $c, C, p, P \geq 0$.
(2) $c \leq C \leq S$ and $p \leq P \leq X$.
Proof: If any of the options has negative value $-v$, buy it for $-v$, i.e., receive the option plus an amount of v in cash, and forget about the option.
If $c > C$, sell a European call option, buy an American call option, pocket $c - C > 0$ and wait for expiration. If the European option is exercised, exercise your American option and deliver the stock.
A similar argument shows that $p \leq P$.

If $C > S$,sell the call option, buy the stock, pocket $C - S > 0$ and wait.
If the option is exercised, deliver your stock, otherwise keep it.
If $P > X$, sell the option and wait. If the option is exercised, buy the stock for X and pocket $P - X > 0$ plus the stock; if the option is not exercised pocket P. $\square$

Proposition 3.2: Assume that the stock does not pay dividends and let r be the T-year spot interest rate. (1) $c > S - Xe^{-rT}$. (2) $p > Xe^{-rT} - S$.

Proof: To prove the first inequality consider:

Portfolio A: one European call option plus an amount of cash equal to Xe^{-rT} deposited for T years at an interest rate of r.

Portfolio B: one share.

After T years portfolio A will yield an amount of cash equal to X.

If, after T years, the stock price S_T is above X, the call option in portfolio A will be exercised, the share sold and the portfolio will be worth S_T. Otherwise, after T years, $S_T \leq X$, the option is not exercised and the portfolio will be worth X.
So after T years portfolio A is worth $\max(S_T, X) \geq S_T$, and since portfolio B is always worth S_T after T years,
the initial value of portfolio A must be no less that the initial value of portfolio B, which is just S.
But since sometimes portfolio A is worth more than portfolio B we have a strict inequality $c + Xe^{-rT} > S$.

To prove $p > Xe^{-rT} - S$ consider:
Portfolio C: one European put option plus one share.
Portfolio D: an amount of cash equal to Xe^{-rT} deposited for T years at an interest rate of r.
After T years portfolio D will be worth X.
If, after T years, $S_T < X$, then the put option in portfolio C will be exercised; the share is sold for X and the portfolio will be worth X. Otherwise, if , after T years, $S_T \geq X$, the option is not exercised and the portfolio will be worth S_T. So after T years portfolio C is worth $\max(S_T, X)$ and the initial value of portfolio C must be no less that the initial value of portfolio D which is just Xe^{-rT}. But since sometimes portfolio C is worth more than portfolio D we have a strict inequality $p + S > Xe^{-rT}$. $\square$

Put-Call Parity

Proposition 3.3 (Put-Call Parity): Assume that the stock does not pay dividends and let r be the T-year spot interest rate. Then $c + Xe^{-rT} = p + S$.

Proof: Recall portfolios A and C:

Portfolio A: one European call option plus an amount of cash equal to Xe^{-rT} deposited for T years at an interest rate of r.

Portfolio C: one European put option plus one share.

After T years they are both worth $\max(S_T, X)$ where S_T is the stock price after T years.

These two portfolios must have identical initial values, i.e., $c + Xe^{-rT} = p + S$. $\square$

Optimal exercise for American style options

When should a rational investor exercise an American option?
Proposition 3.4: Assume that the stock does not pay dividends. The optimal exercise time for the American call option occurs at the expiration of the option and hence $c = C$.
Proof: For any time $0 < \tau < T$ write S_τ, c_τ, C_τ for the values at time τ years of the stock, European option and American option, respectively, and let r be the $T - \tau$ spot interest rate at time τ
$C_\tau \geq c_\tau > S_\tau - Xe^{-r(T-\tau)} > S_\tau - X$, but $S_\tau - X$ is the payoff, at time τ, from the exercise of the American option, and since the value of the American option exceeds that of this payoff, it should not be exercised. The optimal exercise of the American option will produce the same payoff as the European option, hence $c = C$.
□

Optimal exercise of American put options

American put options may have an early optimal exercise date: e.g., suppose that on June 25th, 2002 you held American put options on Worldcom stock with strike price of $65 expiring in September 2002.
Since you bought the stock the company disclosed that it inflated profits for over a year by improperly accounting for more than $3.9 billion and the stock now sells for $0.20.

Great news!

The payoff from exercising your option now would be $64.8, almost its theoretical maximum.
Things can only get worse as time progresses and you should exercise your options now.

Proposition 3.5: $S - X < C - P < S - Xe^{-rT}$.
Proof:
Since $P \geq p$ always, and since some of the time the payoff of the American option will be greater than that of the corresponding European option, $P > p$.
The second inequality is a consequence of Put-Call Parity, $P > p$ and $c = C$.

To prove $S - X < C - P$ consider:
Portfolio E: one European call option plus an amount of cash equal to X deposited for T years at an interest rate of r.
Portfolio F: one American put option plus one share.
At the time of expiration, portfolio E will be worth
$\max(S_T - X, 0) + Xe^{rT} = \max(S_T, X) - X + Xe^{rT} =$
$\max(S_T, X) + X(e^{rT} - 1) > \max(S_T, X)$
and, if the American option has not been exercised before, portfolio F will be worth $\max(X - S_T, 0) + S_T = \max(S_T, X)$. So portfolio E expires with higher value than portfolio F.

If the American option was (rationally) exercised at an earlier time $0 \leq \tau < T$, then at that time portfolio F was worth $(X - S_\tau) + S_\tau = X$. At that time, the value of the cash in portfolio E is at least X. Since in either case there is always a time at which portfolio E is more valuable than portfolio F, the present value of portfolio E is greater than the present value of portfolio F, i.e., $C + X > P + S$. $\square$

Parameters affecting the prices of options

Suppose you hold a call option on a stock whose present price is £10 expiring in T years.
What should happen to the price c of the call option if the stock price goes up to £15?
The payoff of the option at expiration is $\max(S_T - X, 0)$ where S_T is the price of the stock at expiration.
The rise in the stock price suggests that the market expects S_T to be higher as well,
so *c and C are increasing functions of S.* For similar reasons, *p and P are decreasing functions of S.* Obviously, c and C are decreasing functions of X while p and P are increasing functions of X.

Suppose that the strike price of your call option is £20. If you are told that the variability of the stock price is very small, your option is not worth much because when the option expires the stock price is likely to be very close to £10, far below the strike price.
In general, share price movements can result in both higher and lower payoffs from an option, but the downside is limited (to losing the price paid for the option) while the upside is unlimited (in the case of a call option) or bounded by the strike price which is usually much higher than the option price.
So we expect that c, C *are increasing functions of the volatility of the stock price.*

It is reasonable to assume that for longer expiration times T, the value of the stock at time T will have more variability.
But on the other hand, for bigger T, the payoff of the option has to be discounted by a smaller discount factor.
Obviously, *C and P are increasing functions of T.*

The End