Binomial trees and risk neutral valuation

Moty Katzman

September 19, 2014

Derivatives in a simple world

A derivative is an asset whose value depends on the value of another asset.
Call/Put European/American options are examples of derivatives.
We want to find prices of derivatives providing a single payoff at a future date when the underlying asset price evolves in a particularly simple way.

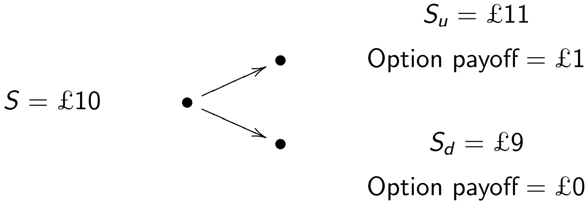
A quick quiz

Consider a European call option on stock whose spot price is £10.
The option expires in one year and has a strike price of £10.
Both Mrs. X and Mr. Y believe that in a year the price of the stock will be either £11 or £9.
Mrs. X believes that the probability of a rise in price is 1/2, while Mr. Y believes that probability is 1/3.
Who is willing to pay more for the option?

Consider a 1-year European call option on a stock with strike price £10.

Assume that the current price of the stock is $S_0 = £10$ and that at the end of the one year period the price of the stock will be either $S_u = £11$ or $S_d = £9$.

Assume further that the 1-year interest rate is 5%.



What should the price c of the option be?
Consider a portfolio with δ shares of this stock, and short in one option.

$$
\begin{array}{lcl}
 & & S_u = £11 \\
 & \nearrow \bullet & \text{Option payoff} = £1 \\
S = £10 \quad \bullet & & \\
 & \searrow \bullet & S_d = £9 \\
 & & \text{Option payoff} = £0
\end{array}
$$

If the stock price goes up the portfolio will be worth $11\delta - 1$ and if the stock price goes down it will be worth 9δ. What if we choose our δ so that $11\delta - 1 = 9\delta, \text{ i.e., } \delta = \frac{1}{2}$?
The value of this portfolio is the same in all possible states of the world!

The portfolio must have a present value equal to its value in one year discounted to the present, i.e., $\frac{9}{2}e^{-0.05\times 1}$ but the current price of stock in the portfolio is £10/2, so

$$\frac{9}{2}e^{-0.05} = 5 - c \Rightarrow c = 5 - \frac{9}{2}e^{-0.05} \approx 0.72.$$

<u>The probability of up or down movements</u>
<u>in the stock price plays no role whatsoever!</u>
(And Mrs. X and Mr. Y will be willing to play the same price for the option.)

We generalise:
consider a financial asset which provides no income and a financial derivative on that asset providing a single payoff t years in the future.
The current price of the asset is S in t years the price of the stock will be either Su ($u > 1$), resulting in a payoff of P_u from the derivative, or Sd ($0 \leq d < 1$) resulting in a payoff of P_d from the derivative.
Let r be the t-year interest rate.

$$
\begin{tikzcd}[row sep=small]
 & & \bullet & \begin{array}{c}\text{asset price} = Su \\ \text{Option payoff} = P_u\end{array} \\
S & \bullet \arrow[ur] \arrow[dr] & & \\
 & & \bullet & \begin{array}{c}\text{asset price} = Sd \\ \text{Option payoff} = P_d\end{array}
\end{tikzcd}
$$

Construct a portfolio consisting of δ units of the asset and -1 units of the derivative.
Choose δ so that the value of the portfolio after t years is certain: δ must satisfy

$$\delta Su - P_u = \delta Sd - P_d \Rightarrow \delta = \frac{P_u - P_d}{S(u-d)}.$$

The value of the portfolio in t years will be

$$\frac{P_u - P_d}{S(u-d)} Su - P_u = \frac{P_u - P_d}{u-d} u - P_u$$

and its present value is

$$e^{-rt} \left(\frac{P_u - P_d}{u-d} u - P_u \right).$$

Let x be the price of the derivative. We must have the following equality of present values $e^{-rt}\left(\frac{P_u - P_d}{u-d}u - P_u\right) = \delta S - x$
$= \frac{P_u - P_d}{S(u-d)}S - x = \frac{P_u - P_d}{u-d} - x$
Solving for x we obtain $x = \frac{P_u - P_d}{u-d} - e^{-rt}\left(\frac{P_u - P_d}{u-d}u - P_u\right) =$
$\frac{e^{-rt}}{u-d}\left((e^{rt}-d)P_u + (u - e^{rt})P_d\right)$ and if we let $q = \frac{e^{rt}-d}{u-d}$ we can rewrite x as $x = e^{-rt}\left(qP_u + (1-q)P_d\right)$.

Notice: $0 \leq q = \frac{e^{rt}-d}{u-d} \leq 1$. We can interpret q as a probability; *In a world where the probability of the up movement in the asset price is q, the equation* $x = e^{-rt}\left(qP_u + (1-q)P_d\right)$ *says that the price of the derivative is the expected present value of its payoff.*
Using these probabilities, stock price at time t has expected value
$$E = qSu + (1-q)Sd = qS(u-d) + Sd = \frac{e^{rt}-d}{u-d}S(u-d) + Sd =$$
$(e^{rt} - d)S + Sd = e^{rt}S$, i.e., *the world where the probability of the up movement in the asset price is q is one in which the stock price grows on average at the risk-free interest rate.*
So in this world investors are indifferent to risk (unlike real-life investors)

We refer to the probabilities q and $1 - q$ as *risk neutral probabilities* and to equation above as a *risk neutral valuation.*

An example

Consider a stock whose current price is £20 and whose price in 3 months will be either £22 or £18. Let c be the price of a European call option on this stock with strike price £20 and expiring in three months. Assume that the 3-month interest rate is 5%.
Let p be the probability of an upward movement in the stock price *in a risk neutral world.*
In such a world the expected price of the stock must be $20e^{0.05/4} = 20e^{1/80}$, so p satisfies
$22p + 18(1-p) = 20e^{1/80} \Rightarrow p = 5e^{1/80} - \dfrac{9}{2} \approx 0.5629$. The expected payoff of the option is now $2p + 0(1-p) = 2p$ and its present value is $2pe^{-0.05/4} \approx 1.112$.

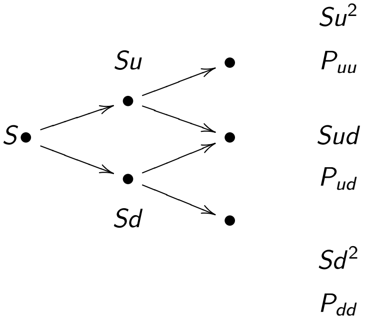
A two step process

Now the price of the underlying changes twice, each time by either a factor of $u > 1$ or $d < 1$.

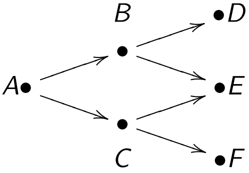
After two periods the stock price will be Su^2, $Sud = Sdu$ or Sd^2.

The derivative expires after the two periods producing payoffs of P_{uu}, $P_{ud} = P_{du}$ and P_{dd} respectively.

Assume also each period is Δt years long and that interest rates for all periods is r.



To find x, the value of the derivative, we now work our way backwards, from the end of the tree (i.e., the end of the second period) to the root (i.e., the present.)



The value of the derivative is known at vertices D,E and F; these are the payoffs P_{uu}, P_{ud} and P_{dd}.
How about nodes B and C?

We can find the value of the derivative at node B by considering the following one period tree:

$$\begin{tikzcd}[row sep=small]
 & & Su^2 \\
 & \bullet & P_{uu} \\
v_B \bullet \arrow[ur] \arrow[dr] & & \\
 & \bullet & Sud \\
 & & P_{ud}
\end{tikzcd}$$

The risk neutral probability q of an upward movement is given by $q = \dfrac{e^{r\Delta t} - d}{u - d}$ and so the value v_B of the derivative at node B is $v_B = e^{-r\Delta t}\left(qP_{uu} + (1-q)P_{ud}\right)$.

Similarly, the value of the derivative at node C is obtained from

$$\begin{tikzcd}[row sep=small]
 & & Sud \\
 & \bullet & P_{ud} \\
v_C \bullet \arrow[ur] \arrow[dr] & & \\
 & \bullet & Sd^2 \\
 & & P_{dd}
\end{tikzcd}$$

$$v_C = e^{-r\Delta t}\left(qP_{ud} + (1-q)P_{dd}\right).$$

Now we can work our way back one more step to node A;

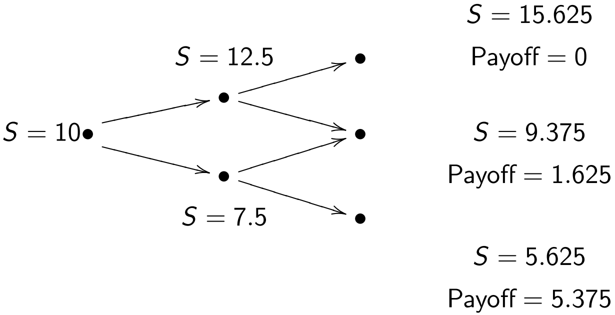
$$\begin{tikzcd}[row sep=tiny]
 & \bullet v_B \\
v_A \bullet \arrow[ur] \arrow[dr] & \\
 & \bullet v_C
\end{tikzcd}$$

and the value at node A is

$$v_A = e^{-r\Delta t}\left(qv_B + (1-q)v_C\right).$$

An example

Consider a European put option on stock currently traded at £10, with strike price £11 and expiring in one year. Interest rates for all periods are 4%.
Use a two 6-month-period tree with $u = 5/4$ and $d = 3/4$ to estimate the price of the option.



Risk neutral probability of an upward movement of stock price

$$q = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.04/2} - 3/4}{5/4 - 3/4} \approx 0.5404.$$

Value v_B of the derivative at node B

$$v_B = e^{-r\Delta t}\left(qP_{uu} + (1-q)P_{ud}\right) \approx 0.7321,$$

value of the derivative at node C

$$v_C = e^{-r\Delta t}\left(qP_{ud} + (1-q)P_{dd}\right) \approx 3.282,$$

value at node A

$$v_A = e^{-r\Delta t}\left(qv_B + (1-q)v_C\right) \approx 1.866$$

n-step trees

Example: Consider a 18-month European put option with strike £12 on a stock whose current price is £10.
Assume that interest rates for all periods are 5%.
Use $u = 6/5$ and $d = 4/5$ to construct the following three step binomial tree.

$$
\begin{tikzcd}[column sep=small, row sep=tiny]
 & & & 17.28\ \ 0 \\
 & & 14.4\ \ 0.2044 \arrow[ur] \arrow[dr] & \\
 & 12\ \ 1.008 \arrow[ur] \arrow[dr] & & 11.52\ \ 0.48 \\
10\ \ 2.008 \arrow[ur] \arrow[dr] & & 9.6\ \ 2.104 \arrow[ur] \arrow[dr] & \\
 & 8\ \ 3.415 \arrow[ur] \arrow[dr] & & 7.68\ \ 4.32 \\
 & & 6.4\ \ 5.304 \arrow[ur] \arrow[dr] & \\
 & & & 5.12\ \ 6.88
\end{tikzcd}
$$

$$q = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05/2} - 4/5}{6/5 - 4/5} \approx 0.5633.$$

$$e^{-0.05/2}(0.5633 \times 0 + 0.4367 \times 0.48) \approx 0.2044$$

$$e^{-0.05/2}(0.5633 \times 0.48 + 0.4367 \times 4.32) \approx 2.104$$

$$e^{-0.05/2}(0.5633 \times 4.32 + 0.4367 \times 6.88) \approx 5.304$$

$$e^{-0.05/2}(0.5633 \times 0.2044 + 0.4367 \times 2.104) \approx 1.008$$

$$e^{-0.05/2}(0.5633 \times 2.104 + 0.4367 \times 5.304) \approx 3.415$$

$$e^{-0.05/2}(0.5633 \times 1.008 + 0.4367 \times 3.415) \approx 2.008$$

Example: an American option

Consider a 18-month American put option with strike £12 on a stock whose current price is £10. Assume that interest rates for all periods are 5%. Use $u = 6/5$ and $d = 4/5$ to construct a three step binomial tree. Consider the “dd” node in the previous figure. Immediate exercise gives payoff of $12 - 6.4 = 5.6 > 5.304$ and that is the value of the option at this node.

$$\begin{tikzcd}[column sep=small] & 7.68\ \ 4.32 \\ 6.4\ \ 5.304 \arrow[ur] \arrow[dr] & \\ & 5.12\ \ 6.88 \end{tikzcd}$$

The modified tree for the American option is then

$$\begin{tikzcd}
 & & & 17.28\ 0 \\
 & & 14.4\ 0.2044 \arrow[ur] \arrow[dr] & \\
 & 12\ \underline{1.1345} \arrow[ur] \arrow[dr] & & 11.52\ 0.48 \\
10\ \underline{2.327} \arrow[ur] \arrow[dr] & & 9.6\ \underline{2.4} \arrow[ur] \arrow[dr] & \\
 & 8\ \underline{4} \arrow[ur] \arrow[dr] & & 7.68\ 4.32 \\
 & & 6.4\ \underline{5.6} \arrow[ur] \arrow[dr] & \\
 & & & 5.12\ 6.88
\end{tikzcd}$$

Underlined values differ from the European style case.

$$q = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05/2} - 4/5}{6/5 - 4/5} \approx 0.5633.$$

$$e^{-0.05/2}(0.5633 \times 0 + 0.4367 \times 0.48) \approx 0.2044$$

$$e^{-0.05/2}(0.5633 \times 0.48 + 0.4367 \times 4.32) \approx 2.104, 12 - 9.6 = 2.4 > 2.104$$

$$e^{-0.05/2}(0.5633 \times 4.32 + 0.4367 \times 6.88) \approx 5.304, 12 - 6.4 = 5.6 > 5.304$$

$$e^{-0.05/2}(0.5633 \times 0.2044 + 0.4367 \times 2.4) \approx 1.1345$$

Using binomial trees for approximating values of derivatives

The assumption that the price of the asset underlying a derivative changes at a finite number of moments can approximate reality only if we allow the price to change at a large number of points in time, many more than two or three.

This can lead to n-step trees for large values of n.

These will contain $1 + 2 + 3 + \cdots + n = n(n+1)/2$ nodes and for even a modest value of n, say $n = 10$, these computations are best left to computers.

In the case of certain exotic derivatives their value depends not only on the final price of an asset but on its history as well. These derivatives require even larger binomial trees.

Which values of u and d shall we use?

Different choices result in different prices for derivatives!
Common practice: $d = 1/u$ and $u = e^{\sigma\sqrt{\Delta t}}$ (σ is the yearly standard deviation of the logarithm of the stock price and Δt is the length in years of every step in the tree.
(In the next chapter we will adopt a model for the evolution of stock prices which implies that the logarithm of stock prices Δt years in the future are normally distributed random variables with standard deviation $\sigma\sqrt{\Delta t}$.

With our choice $\log S$ increases by $\sigma\sqrt{\Delta t}$ with probability $q = \frac{e^{r\Delta t}-d}{u-d}$ or decreases by $\sigma\sqrt{\Delta t}$ with probability $1-q$. This recovers the variance of the logarithm of the stock price.

The End