The stochastic process followed by stock prices.

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The price of a certain stock at a future time $t$ is unknown at the present.
We think of it as being a random variable $S_t$.

What sort of random variable are $S_t$ for $t \geq 0$?
Brownian motion

A Brownian motion is a family of random variables

\[ \{B_t | t \geq 0\} \]

on some probability space \((\Omega, \mathcal{F}, P)\) such that:

1. \(B_0 = 0\),
2. for \(0 \leq s < t\) the increment \(B_t - B_s\) is normally distributed
   with mean 0 and variance \(t - s\),
3. for any \(0 \leq t_1 < t_2 < \cdots < t_n\) the increments
   \[ B_{t_1} - B_0, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}} \]
   are independent random variables, and
4. For any \(\omega \in \Omega\) the function \(t \mapsto B_t(\omega)\) is continuous.
Three instances of Brownian Motion corresponding to $\omega_1, \omega_2, \omega_3 \in \Omega$. 
Does Brownian motion exist?
Yes– by Kolmogorov’s Existence Theorem.
Brownian motion has surprising properties; for example,
(1) The function \( t \mapsto B_t(\omega) \) is nowhere differentiable with probability 1,
(2) if \( B_t = x \) for some \( t \) then for any \( \epsilon > 0 \) the set
\( \{ \tau : |\tau - t| < \epsilon \text{ and } B_\tau = x \} \) is infinite with probability one.

Brownian motion is useful for describing the jiggling of prices: buying and selling jiggle prices.
Brownian motion is an example of a *stochastic process* i.e., a family of random variables indexed by time $t \geq 0$. We now construct a more general kind of stochastic processes whose definition is based on Brownian motion.
We want to construct a stochastic process \( \{ X_t \mid t \geq 0 \} \) on the same probability space \((\Omega, \mathcal{F}, P)\) on which the Brownian motion \( \{ B_t \mid t \geq 0 \} \) is defined with the property that the change of \( X \) over an infinitesimal period of time \( dt \) is given by

\[
dX = a(\omega, t)dt + b(\omega, t)dB
\]

where \( a \) and \( b \) are themselves stochastic processes on \((\Omega, \mathcal{F}, P)\) with continuous paths and where \( dB \) is the change in the Brownian motion over the infinitesimal period of time \( dt \).
Fix any $\omega \in \Omega$; for any partition $\mathcal{P}$ of $[0, t]$ into small intervals $[s_0, s_1], [s_1, s_2], \ldots, [s_{n-1}, s_n]$ where $s_0 = 0$ and $s_n = t$, we compute the sum
\[
\Sigma_\mathcal{P} = \sum_{i=0}^{n-1} b(\omega, s_i) (B(\omega)_{s_{i+1}} - B(\omega)_{s_i}) .
\]
Define the *norm* of the partition $\mathcal{P}$ to be
\[
\|\mathcal{P}\| = \max \{ s_1 - s_0, s_2 - s_1, \ldots, s_n - s_{n-1} \} ;
\]
If $b$ satisfies some technical conditions, the limit as $\|\mathcal{P}\| \to 0$ of $\Sigma_\mathcal{P}$ exists. This limit is known as an *Ito integral* and we denote it with
\[
\int_0^t b(\omega, s) dB(\omega)_s .
\]
We now define the process
\[
X_t(\omega) = X_0(\omega) + \int_0^t a(\omega, s) ds + \int_0^t b(\omega, s) dB(\omega)_s
\]
for all $t \geq 0$. 
Example: \[ \int_0^t s \, dB_s \]

We compute the integral from definition

(1) Take any partition \( \mathcal{P} \) of \([0, t]\) into small intervals \([s_0, s_1], [s_1, s_2], \ldots, [s_{n-1}, s_n]\) where \( s_0 = 0 \) and \( s_n = t \).

(2) The sum

\[
\sum_{\mathcal{P}} = \sum_{i=0}^{n-1} s_i \left(B_{s_{i+1}} - B_{s_i}\right)
\]

is a sum of independent normally distributed random variables with mean 0 and variance \( s_0^2(s_1 - s_0), s_1^2(s_2 - s_1), \ldots, s_{n-1}^2(s_n - s_{n-1}) \).

(3) \( \sum_{\mathcal{P}} \) is normally distributed with mean 0 and variance \( s_0^2(s_1 - s_0) + s_1^2(s_2 - s_1) + \cdots + s_{n-1}^2(s_n - s_{n-1}) \).

(4) As \( \|\mathcal{P}\| \to 0 \) this variance converges to \( \int_0^t s^2 \, ds = t^3/3 \).

(5) We conclude that \( \int_0^t s \, dB_s \) is a normally distributed random variable with mean 0 and variance \( t^3/3 \).
Henceforth we write

\[ dX = a(X, t) \, dt + b(X, t) \, dB \]

to denote that fact that \( X \) is a stochastic process defined by

\[ X_t(\omega) = X_0(\omega) + \int_0^t a(X_s(\omega), s) \, ds + \int_0^t b(X_s(\omega), s) \, dB(\omega)_s. \]

We shall refer to stochastic processes of this form \textit{Ito processes}. 
As a first approximation we model the proportional increase in stock prices as a Brownian motion. We could then derive the following discrete time version

\[
\frac{dS}{S} = \sigma dB
\]

where \(dS\) is the change in the stock price over a short time from \(t\) to \(t + dt\), \(dB = B_{t+dt} - B_{dt}\) and \(B\) is a Brownian motion. (In particular proportional increases in \(S\) are independent, e.g., today’s increase in a stock price is independent of tomorrow’s increase.)
model implies that the values of stocks vary without any long term trend, i.e., \( E \left( \frac{dS}{S} \right) = 0 \).
A quick glance at historical data shows that this is not very plausible:

Despite the randomness of the value of the DJIA, its long term exponential growth is quite visible.
To take into account this upward trend in stock prices we introduce a drift term

\[ \frac{dS}{S} = \sigma dB + \mu dt. \]

We refer to a process \( S \) defined above as a geometric Brownian motion.

Now

\[ E\left( \frac{dS}{S} \right) = \mu dt. \]

We shall refer to \( \mu \) as the expected return of the stock and to \( \sigma \) as the volatility of the stock.
Instances of the process $dS = 0.1Sdt + \sigma SdB$
One can raise several objections to this model. Consider the following list of all trades in Vodafone shares between 16:22:08 and 16:23:08 on August 16th, 2002:
<table>
<thead>
<tr>
<th>Trade time</th>
<th>Trade price</th>
<th>Bid</th>
<th>Ask</th>
<th>Volume</th>
<th>Block price</th>
<th>Buy/Sell</th>
</tr>
</thead>
<tbody>
<tr>
<td>16:23:08</td>
<td>100.75p</td>
<td>100.75p</td>
<td>101p</td>
<td>180,614</td>
<td>£181,969</td>
<td>SELL</td>
</tr>
<tr>
<td>16:23:08</td>
<td>100.75p</td>
<td>100.75p</td>
<td>101p</td>
<td>1,185</td>
<td>£1,194</td>
<td>SELL</td>
</tr>
<tr>
<td>16:23:08</td>
<td>100.75p</td>
<td>100.5p</td>
<td>100.75p</td>
<td>250,000</td>
<td>£251,875</td>
<td>BUY</td>
</tr>
<tr>
<td>16:23:01</td>
<td>100.5p</td>
<td>100.25p</td>
<td>100.75p</td>
<td>2,500</td>
<td>£2,512</td>
<td>BUY</td>
</tr>
<tr>
<td>16:22:55</td>
<td>100.75p</td>
<td>100.5p</td>
<td>100.75p</td>
<td>28,383</td>
<td>£28,596</td>
<td>BUY</td>
</tr>
<tr>
<td>16:22:55</td>
<td>100.75p</td>
<td>100.5p</td>
<td>100.75p</td>
<td>3,000</td>
<td>£3,022</td>
<td>BUY</td>
</tr>
<tr>
<td>16:22:55</td>
<td>100.75p</td>
<td>100.5p</td>
<td>100.75p</td>
<td>248,815</td>
<td>£250,681</td>
<td>BUY</td>
</tr>
<tr>
<td>16:22:55</td>
<td>100.75p</td>
<td>100.5p</td>
<td>100.75p</td>
<td>179,802</td>
<td>£181,151</td>
<td>BUY</td>
</tr>
<tr>
<td>16:22:55</td>
<td>100.75p</td>
<td>100.5p</td>
<td>100.75p</td>
<td>40,000</td>
<td>£40,300</td>
<td>BUY</td>
</tr>
<tr>
<td>16:22:42</td>
<td>100.5p</td>
<td>100.5p</td>
<td>100.75p</td>
<td>1,500</td>
<td>£1,508</td>
<td>SELL</td>
</tr>
<tr>
<td>16:22:41</td>
<td>100.75p</td>
<td>100.5p</td>
<td>100.75p</td>
<td>23,117</td>
<td>£23,290</td>
<td>BUY</td>
</tr>
<tr>
<td>16:22:39</td>
<td>100.527p</td>
<td>100.5p</td>
<td>100.75p</td>
<td>10,000</td>
<td>£10,053</td>
<td>SELL</td>
</tr>
<tr>
<td>16:22:38</td>
<td>100.75p</td>
<td>100.5p</td>
<td>100.75p</td>
<td>48,500</td>
<td>£48,864</td>
<td>BUY</td>
</tr>
<tr>
<td>16:22:38</td>
<td>100.75p</td>
<td>100.5p</td>
<td>100.75p</td>
<td>150,000</td>
<td>£151,125</td>
<td>BUY</td>
</tr>
<tr>
<td>16:22:38</td>
<td>100.75p</td>
<td>100.5p</td>
<td>100.75p</td>
<td>25,000</td>
<td>£25,188</td>
<td>BUY</td>
</tr>
<tr>
<td>16:22:38</td>
<td>100.75p</td>
<td>100.5p</td>
<td>100.75p</td>
<td>26,500</td>
<td>£26,699</td>
<td>BUY</td>
</tr>
<tr>
<td>16:22:29</td>
<td>100.688p</td>
<td>100.5p</td>
<td>100.75p</td>
<td>25,000</td>
<td>£25,172</td>
<td>BUY</td>
</tr>
<tr>
<td>16:22:18</td>
<td>100.75p</td>
<td>100.5p</td>
<td>100.75p</td>
<td>25,000</td>
<td>£25,188</td>
<td>BUY</td>
</tr>
<tr>
<td>16:22:08</td>
<td>100.75p</td>
<td>100.5p</td>
<td>100.75p</td>
<td>11,000</td>
<td>£11,082</td>
<td>BUY</td>
</tr>
</tbody>
</table>
(1) The model allows any real number to be a value for $S$, but in real life there is a smallest unit.
(2) We assume that prices are changing continuously but trades occur at discrete times.
(3) Shares are often traded through market makers. They buy at the bid price and sell at the ask price. So there are two prices! Sometimes it is crucial to model both.
(4) Sometimes, e.g., during market crashes, changes seem to have a “memory”. We should regard our model as an approximation to real life prices and trades, not as an accurate description.
Ito’s Lemma.

Consider a stochastic process $X_t$ whose change over a small interval of time from $t$ to $t + dt$ is given by

$$dX = a(X, t)dt + b(X, t)dB$$

where $a(x, t)$ and $b(x, t)$ are functions of $x$ and $t$.

Consider a new stochastic process $Y = G(X, t)$ where $G(x, t)$ is a function of $x$ and $t$.

What sort of process is $Y$?

**Theorem: (Ito’s Lemma)**

Assume that $G(x, t)$ is twice continuously differentiable with respect to $x$ and continuously differentiable with respect to $t$.

The process $Y$ is also an Ito process.

In fact,

$$dY = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b dB.$$
Write $dB = B_{t+\Delta t} - B_t$ as $\sqrt{\Delta t} Z$ for some standard normal $Z$.
Note: $1 = \text{Var}(Z) = E(Z^2) - E(Z)^2 = E(Z^2)$.
So, $E(dB^2) = E(\Delta t Z^2) = \Delta t E(Z^2) = \Delta t$, and
$\text{Var}(dB^2) = \Delta t^2 \text{Var}(Z^2)$ is of order $\Delta t^2$.
As $\Delta t \to 0$, $\text{Var}(dB^2) \ll \Delta t = E(dB^2)$, so $dB^2$ “converges to” $\Delta t$. 

"$dB^2 = dt$"
(Very) informal proof of Ito’s Lemma:

Using the Taylor series for $G$ we can write

$$dY = \frac{\partial G}{\partial x} \Delta X + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \Delta X^2 + \frac{\partial^2 G}{\partial x \partial t} \Delta X \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} \Delta t^2 + \text{higher order terms.}$$

Now

$$(\Delta X)^2 = (a \Delta t + b dB)^2 = b^2 \Delta t + \text{higher terms in } \Delta t$$
So for small $\Delta t$ we approximate

$$dY \approx$$

$$\frac{\partial G}{\partial x} \Delta X + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \Delta t +$$

higher degree terms in $\Delta X, \Delta t =$

$$\frac{\partial G}{\partial x} (a \Delta t + b \Delta B) + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \Delta t +$$

higher degree terms in $\Delta B, \Delta t$. 
An Example:

Use Ito’s Lemma to show that

\[ d \left( B_t^2 \right) = dt + 2B_t (dB_t). \]

Deduce that

\[ \int_0^t B_s dB_s = \frac{1}{2} B_t^2 - \frac{t}{2}. \]

Let \( X_t \) be the process with \( dX = 0 \times dt + 1 \times dB_t \) and consider the process \( Y_t = G(X_t, t) \) where \( G(x, t) = x^2 \). We have

\[ \frac{\partial G}{\partial x} = 2x, \quad \frac{\partial^2 G}{\partial x^2} = 2, \quad \frac{\partial G}{\partial t} = 0 \]

so Ito’s Lemma implies that

\[ d(B_t^2) = dY = \left( 2X \times 0 + 0 + \frac{1}{2} 2 \times 1^2 \right) dt + 2X \times 1 \ dB = dt + 2B_t \ dB. \]

Integrating both sides of the equation above gives

\[ \int_0^t d(B_s^2) = t + 2 \int_0^t B_s dB_s \Rightarrow \int_0^t B_s dB_s = \frac{1}{2} B_t^2 - \frac{t}{2}. \]
The stochastic process followed by forward stock prices

Consider a forward contract on stock paying no dividends maturing at time $T$; let $F(t)$ be its forward price at time $t \geq 0$:

$$F(t) = S(t)e^{r(T-t)},$$

where $S(t)$ is the spot price of the stock at time $t$.

Regard $F$ as a function of $s$ and $t$, i.e., $F = F(s, t) = se^{r(T-t)}$:

$$\frac{\partial F}{\partial s} = e^{r(T-t)}, \quad \frac{\partial^2 F}{\partial s^2} = 0 \quad \text{and} \quad \frac{\partial F}{\partial t} = -rse^{r(T-t)}.$$

Our model assumes that

$$dS = \mu Sdt + \sigma SdB$$

so Ito’s Lemma implies that

$$dF = \left( e^{r(T-t)} \mu S - rSe^{r(T-t)} \right) dt + e^{r(T-t)} \sigma SdB =$$

$$(\mu - r)Fdt + \sigma FdB,$$

i.e., $F$ follows a geometric Brownian motion with drift $\mu - r$. 
The stochastic process followed by the logarithm of stock prices

Let $S$ be the spot price of a certain stock at time $t$ and let $G = G(s, t) = \log s$. Since

$$\frac{\partial G}{\partial s} = \frac{1}{s}, \quad \frac{\partial^2 G}{\partial s^2} = \frac{-1}{s^2} \quad \text{and} \quad \frac{\partial G}{\partial t} = 0$$

and since $dS = \mu S dt + \sigma S dB$ Ito’s Lemma implies that

$$dG = \left(\frac{1}{S} \mu S - \frac{\sigma^2 S^2}{2S^2}\right) dt + \frac{\sigma S}{S} dB =$$

$$\left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dB.$$
Consider a fixed time $T$ in the future and the spot price $S_T$ at that time: we see that the logarithm of the proportional price change of the stock,

$$\log \frac{S_T}{S} = \log S_T - \log S,$$

is normally distributed

with expected value $\left( \mu - \frac{\sigma^2}{2} \right) T$

and variance $\sigma^2 T$. 

Example:

Consider the price $S$ of a stock with current price $S_0 = $10, expected annual return of $\mu = 15\%$ and annual price volatility (i.e., standard deviation) of $\sigma = 20\%$.

The stock price follows a geometric Brownian motion

$$dS = \mu S \, dt + \sigma S \, dB = 0.15S \, dt + 0.2S \, dB$$

and for any time $T$ in the future the price of the stock at time $T$ satisfies \( \log S_T \sim N \left( \log S_0 + \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma^2 T \right) \) = \( N \left( \log 10 + 0.13T, 0.2^2T \right) \). So for example, one can say that with 90\% confidence in 6 months

$$\log 10 + 0.065 - \frac{0.2}{\sqrt{2}} \times 1.645 < \log S_{1/2} <$$

$$\log 10 + 0.065 + \frac{0.2}{\sqrt{2}} \times 1.645$$

i.e., with 90\% confidence the logarithm of stock price in 6 months will be between approximately 2.135 and 2.6 and so the stock price itself will be between approximately $8.46 and $13.47 with 90\% confidence.
The End