The stochastic process followed by stock prices.

Moty Katzman

September 19, 2014

The stochastic process followed by stock prices

The price of a certain stock at a future time t is unknown at the present.
We think of it as being a random variable S_t.

What sort of random variable are S_t for $t \geq 0$?

Brownian motion

A *Brownian motion* is a family of random variables

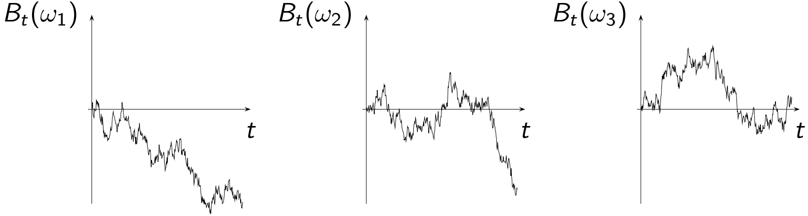
$$\{B_t \mid t \geq 0\}$$

on some probability space $(\Omega, \mathcal{F}, P)$ such that:
(1) $B_0 = 0$,
(2) for $0 \leq s < t$ the increment $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$,
(3) for any $0 \leq t_1 < t_2 < \cdots < t_n$ the increments

$$B_{t_1} - B_0, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$$

are independent random variables, and
(4) For any $\omega \in \Omega$ the function $t \mapsto B_t(\omega)$ is continuous.

Three instances of Brownian Motion corresponding to $\omega_1, \omega_2, \omega_3 \in \Omega$.



Does Brownian motion exist?
Yes– by *Kolmogorov's Existence Theorem.*
Brownian motion has surprising properties; for example,
(1) The function $t \mapsto B_t(\omega)$ is nowhere differentiable with probability 1,
(2) if $B_t = x$ for some t then for any $\epsilon > 0$ the set $\{\tau : |\tau - t| < \epsilon \text{ and } B_\tau = x\}$ is infinite with probability one.

Brownian motion is useful for describing the jiggling of prices: buying and selling jiggle prices.

The Ito integral

Brownian motion is an example of a *stochastic process* i.e., a family of random variables indexed by time $t \geq 0$.
We now construct a more general kind of stochastic processes whose definition is based on Brownian motion.

We want to construct a stochastic process $\{X_t \mid t \geq 0\}$ on the same probability space $(\Omega, \mathcal{F}, \mathrm{P})$ on which the Brownia motion $\{B_t \mid t \geq 0\}$ is defined
with the property that the change of X over an infinitesimal period of time dt is given by

$$dX = a(\omega, t)dt + b(\omega, t)dB$$

where a and b are themselves stochastic processes on $(\Omega, \mathcal{F}, \mathrm{P})$ with continuous paths and where dB is the change in the Brownian motion over the infinitesimal period of time dt.

Fix any $\omega \in \Omega$; for any partition $\mathcal{P}$ of $[0, t]$ into small intervals $[s_0, s_1], [s_1, s_2], \dots, [s_{n-1}, s_n]$ where $s_0 = 0$ and $s_n = t$, we compute the sum

$$\Sigma_{\mathcal{P}} = \sum_{i=0}^{n-1} b(\omega, s_i) \left(B(\omega)_{s_{i+1}} - B(\omega)_{s_i} \right) .$$

Define the *norm* of the partition $\mathcal{P}$ to be

$$\|\mathcal{P}\| = \max \{ s_1 - s_0, s_2 - s_1, \dots, s_n - s_{n-1} \} ;$$

If b satisfies some technical conditions, the limit as $\|\mathcal{P}\| \to 0$ of $\Sigma_{\mathcal{P}}$ exists.
This limit is known as an *Ito integral* and we denote it with

$$\int_0^t b(\omega, s)\, dB(\omega)_s.$$

We now define the process

$$X_t(\omega) = X_0(\omega) + \int_0^t a(\omega, s)\, ds + \int_0^t b(\omega, s)\, dB(\omega)_s$$

for all $t \geq 0$.

Example: $\int_0^t s\, dB_s$

We compute the integral from definition
(1) Take any partition $\mathcal{P}$ of $[0, t]$ into small intervals $[s_0, s_1], [s_1, s_2], \ldots, [s_{n-1}, s_n]$ where $s_0 = 0$ and $s_n = t$.
(2) The sum

$$\Sigma_{\mathcal{P}} = \sum_{i=0}^{n-1} s_i \left(B_{s_{i+1}} - B_{s_i} \right)$$

is a sum of independent normally distributed random variables with mean 0 and variance $s_0^2(s_1 - s_0), s_1^2(s_2 - s_1), \ldots, s_{n-1}^2(s_n - s_{n-1})$.
(3) $\Sigma_{\mathcal{P}}$ is normally distributed with mean 0 and variance $s_0^2(s_1 - s_0) + s_1^2(s_2 - s_1) + \cdots + s_{n-1}^2(s_n - s_{n-1})$.
(4) As $\|\mathcal{P}\| \to 0$ this variance converges to $\int_0^t s^2 ds = t^3/3$.
(5) We conclude that $\int_0^t s dB_s$ is a normally distributed random variable with mean 0 and variance $t^3/3$.

Henceforth we write

$$dX = a(X,t)\,dt + b(X,t)\,dB$$

to denote that fact that X is a stochastic process defined by

$$X_t(\omega) = X_0(\omega) + \int_0^t a(X_s(\omega), s)\,ds + \int_0^t b(X_s(\omega), s)\,dB(\omega)_s.$$

We shall refer to stochastic processes of this form *Ito processes.*

Back to stock prices

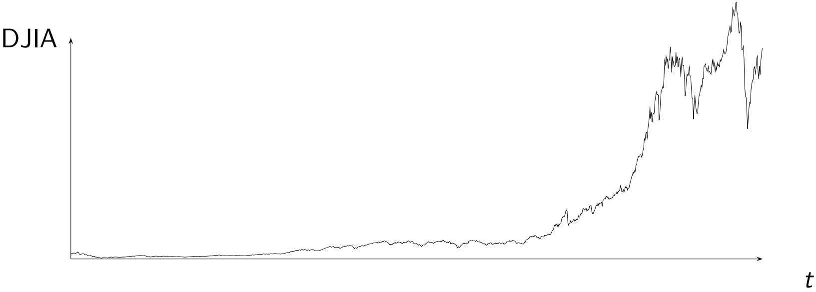
As a first approximation we model the proportional increase in stock prices as a Brownian motion.
We could then derive the following discrete time version

$$\frac{dS}{S} = \sigma dB$$

where dS is the change in the stock price over a short time from t to $t + dt$, $dB = B_{t+dt} - B_{dt}$ and B is a Brownian motion.
(In particular proportional increases in S are independent, e.g., today's increase in a stock price is independent of tomorrow's increase.)

model implies that the values of stocks vary without any long term trend, i.e., $E(\frac{dS}{S}) = 0$.

A quick glance at historical data shows that this is not very plausible:



Despite the randomness of the value of the DJIA, its long term exponential growth is quite visible.

To take into account this upward trend in stock prices we introduce a *drift term*

$$\frac{dS}{S} = \sigma dB + \mu \mathbf{dt}.$$

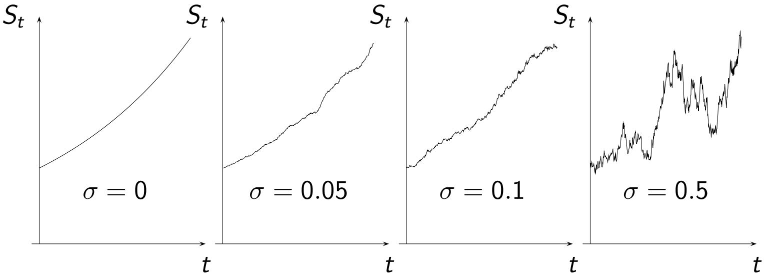
We refer to a process S defined above as a *geometric Brownian motion.*

Now

$$E(\frac{dS}{S}) = \mu dt.$$

We shall refer to μ as the *expected return* of the stock and to σ as the *volatility* of the stock.

Instances of the process $dS = 0.1Sdt + \sigma SdB$



One can raise several objections to this model. Consider the following list of all trades in Vodafone shares between 16:22:08 and 16:23:08 on August 16th, 2002:

Trade time	Trade price	Bid	Ask	Volume	Block price	Buy/Sell
16:23:08	100.75p	100.75p	101p	180,614	£181,969	SELL
16:23:08	100.75p	100.75p	101p	1,185	£1,194	SELL
16:23:08	100.75p	100.5p	100.75p	250,000	£251,875	BUY
16:23:01	100.5p	100.25p	100.75p	2,500	£2,512	
16:22:55	100.75p	100.5p	100.75p	28,383	£28,596	BUY
16:22:55	100.75p	100.5p	100.75p	3,000	£3,022	BUY
16:22:55	100.75p	100.5p	100.75p	248,815	£250,681	BUY
16:22:55	100.75p	100.5p	100.75p	179,802	£181,151	BUY
16:22:55	100.75p	100.5p	100.75p	40,000	£40,300	BUY
16:22:42	100.5p	100.5p	100.75p	1,500	£1,508	SELL
16:22:41	100.75p	100.5p	100.75p	23,117	£23,290	BUY
16:22:39	100.527p	100.5p	100.75p	10,000	£10,053	SELL
16:22:38	100.75p	100.5p	100.75p	48,500	£48,864	BUY
16:22:38	100.75p	100.5p	100.75p	150,000	£151,125	BUY
16:22:38	100.75p	100.5p	100.75p	25,000	£25,188	BUY
16:22:38	100.75p	100.5p	100.75p	26,500	£26,699	BUY
16:22:29	100.688p	100.5p	100.75p	25,000	£25,172	BUY
16:22:18	100.75p	100.5p	100.75p	25,000	£25,188	BUY
16:22:08	100.75p	100.5p	100.75p	11,000	£11,082	BUY

(1) The model allows any real number to be a value for S, but in real life there is a smallest unit.
(2) We assume that prices are changing continuously but trades occur at discrete times.
(3) Shares are often traded through market makers.
They buy at the *bid* price and sell at the *ask* price. So there are two prices! Sometimes it is crucial to model both.
(4) Sometimes, e.g., during market crashes, changes seem to have a "memory".
We should regard our model as an approximation to real life prices and trades, not as an accurate description.

Ito's Lemma.

Consider a stochastic process X_t whose change over a small interval of time from t to $t + dt$ is given by

$$dX = a(X, t)dt + b(X, t)dB$$

where $a(x, t)$ and $b(x, t)$ are functions of x and t.
Consider a new stochastic process $Y = G(X, t)$ where $G(x, t)$ is a function of x and t.
What sort of process is Y?
Theorem: (Ito's Lemma)
Assume that $G(x, t)$ is twice continuously differentiable with respect to x and continuously differentiable with respect to t.
The process Y is also an Ito process.
In fact,

$$dY = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b \; dB.$$

"$dB^2 = dt$"

Write $dB = B_{t+\Delta t} - B_t$ as $\sqrt{\Delta t}Z$ for some standard normal Z.
Note: $1 = \mathrm{Var}(Z) = \mathrm{E}(Z^2) - \mathrm{E}(Z)^2 = \mathrm{E}(Z^2)$.
So, $\mathrm{E}(dB^2) = \mathrm{E}(\Delta t Z^2) = \Delta t\,\mathrm{E}(Z^2) = \Delta t$, and
$\mathrm{Var}(dB^2) = \Delta t^2\,\mathrm{Var}(Z^2)$ is of order Δt^2.
As $\Delta t \to 0$, $\mathrm{Var}(dB^2) \ll \Delta t = \mathrm{E}(dB^2)$, so dB^2 "converges to" Δt.

(Very) informal proof of Ito's Lemma:

Using the Taylor series for G we can write

$$dY = \frac{\partial G}{\partial x}\Delta X + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta X^2 + \frac{\partial^2 G}{\partial x \partial t}\Delta X \Delta t +$$

$$\frac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \text{ higher order terms.}$$

Now

$$(\Delta X)^2 = (a\,\Delta t + b dB)^2 = b^2 \Delta t + \text{ higher terms in } \Delta t$$

So for small Δt we approximate

$$dY \approx$$

$$\frac{\partial G}{\partial x}\Delta X + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\Delta t +$$

higher degree terms in $\Delta X, \Delta t =$

$$\frac{\partial G}{\partial x}\left(a\Delta t + b\Delta B\right) + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\Delta t +$$

higher degree terms in $\Delta B, \Delta t$.

An Example:

Use Ito's Lemma to show that

$$d\left(B_t^2\right) = dt + 2B_t(d\,B_t).$$

Deduce that

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{t}{2}.$$

Let X_t be the process with $dX = 0 \times dt + 1 \times dB_t$ and consider the process $Y_t = G(X_t, t)$ where $G(x,t) = x^2$. We have

$$\frac{\partial G}{\partial x} = 2x, \quad \frac{\partial^2 G}{\partial x^2} = 2, \quad \frac{\partial G}{\partial t} = 0$$

so Ito's Lemma implies that $d(B_t^2) = dY =$

$$\left(2X \times 0 + 0 + \frac{1}{2}2 \times 1^2\right) dt + 2X \times 1\ dB = dt + 2B_t dB.$$

Integrating both sides of the equation above gives

$$\int_0^t d\left(B_s^2\right) = t + 2\int_0^t B_s dB_s \Rightarrow \int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{t}{2}.$$

The stochastic process followed by forward stock prices

Consider a forward contract on stock paying no dividends maturing at time T; let $F(t)$ be its forward price at time $t \geq 0$:

$$F(t) = S(t)e^{r(T-t)},$$

where $S(t)$ is the spot price of the stock at time t.
Regard F as a function of s and t, i.e., $F = F(s,t) = se^{r(T-t)}$:

$$\frac{\partial F}{\partial s} = e^{r(T-t)}, \quad \frac{\partial^2 F}{\partial s^2} = 0 \quad \text{and} \quad \frac{\partial F}{\partial t} = -rse^{r(T-t)}.$$

Our model assumes that

$$dS = \mu S dt + \sigma S dB$$

so Ito's Lemma implies that

$$dF = \left(e^{r(T-t)}\mu S - rSe^{r(T-t)} \right) dt + e^{r(T-t)}\sigma S dB =$$

$$(\mu - r)F dt + \sigma F dB,$$

i.e., F follows a geometric Brownian motion with drift $\mu - r$.

The stochastic process followed by the logarithm of stock prices

Let S be the spot price of a certain stock at time t and let $G = G(s,t) = \log s$. Since

$$\frac{\partial G}{\partial s} = \frac{1}{s}, \quad \frac{\partial^2 G}{\partial s^2} = \frac{-1}{s^2} \quad \text{and} \quad \frac{\partial G}{\partial t} = 0$$

and since $dS = \mu S dt + \sigma S dB$ Ito's Lemma implies that

$$dG = \left(\frac{1}{S} \mu S - \frac{\sigma^2 S^2}{2S^2} \right) dt + \frac{\sigma S}{S} dB =$$

$$\left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dB.$$

Consider a fixed time T in the future and the spot price S_T at that time: we see that the logarithm of the proportional price change of the stock,

$$\log \frac{S_T}{S} = \log S_T - \log S,$$

is normally distributed

with expected value $\left(\mu - \frac{\sigma^2}{2}\right) T$

and variance $\sigma^2 T$.

Example:

Consider the price S of a stock with current price $S_0 = \$10$, expected annual return of $\mu = 15\%$ and annual price volatility (i.e., standard deviation) of $\sigma = 20\%$.

The stock price follows a geometric Brownian motion

$$dS = \mu S\,dt + \sigma S\,dB = 0.15S\,dt + 0.2S\,dB$$

and for any time T in the future the price of the stock at time T satisfies $\log S_T \sim N\left(\log S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right) = N\left(\log 10 + 0.13T, 0.2^2 T\right)$. So for example, one can say that with 90% confidence in 6 months

$$\log 10 + 0.065 - \frac{0.2}{\sqrt{2}} \times 1.645 < \log S_{1/2} <$$

$$\log 10 + 0.065 + \frac{0.2}{\sqrt{2}} \times 1.645$$

i.e., with 90% confidence the logarithm of stock price in 6 months will be between approximately 2.135 and 2.6 and so the stock price itself will be between approximately \$8.46 and \$13.47 with 90% confidence.

The End