The Black-Scholes pricing formulas

Moty Katzman

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The Black-Scholes differential equation

Aim: Find a formula for the
price of European options on stock.

Lemma 6.1:

Assume that a stock price S follows the Geometric Brownian motion

$$dS = \mu S \, dt + \sigma S \, dB$$

where μ and σ are constants.

Let $f = f(S, t)$ be the value at time t of any derivative contingent on the value of S at some $t = T$.

Assume $f(s, t)$ is twice differentiable with respect to s and differentiable with respect to t. The process followed by f is

$$df = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dB.$$

Proof:
Apply Ito's Lemma with $a(S,t) = \mu S$ and $b(S,t) = \sigma S$. ■
The discrete version of the equation is

$$\Delta f = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \Delta B.$$

where ΔB is a normally distributed random variable with zero mean and variance Δt
Consider a portfolio consisting of a variable quantity $\frac{\partial f}{\partial S}$ of shares and -1 derivatives; let Π be the value of this portfolio, i.e.,
$\Pi = \frac{\partial f}{\partial S} S - f$.

After a short period of time Δt the value of the portfolio changes by

$$\Delta \Pi = \frac{\partial f}{\partial S}\Delta S - \Delta f$$

$$= \frac{\partial f}{\partial S}\left(\mu S \Delta t + \sigma S \Delta B\right) - \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t - \frac{\partial f}{\partial S}\sigma S \Delta B$$

$$= \left(-\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 - \frac{\partial f}{\partial t}\right)\Delta t$$

Notice that $\Delta \Pi$ is non-stochastic!

Since the value of Π in the future is known with certainty, its value must be increasing at the same rate as a risk-free deposit earning interest r:

$$\Delta\Pi = e^{r\Delta t}\Pi - \Pi$$

and for infinitesimal Δt we obtain

$$\Delta\Pi \approx (1 + r\Delta t)\Pi - \Pi \Rightarrow \Delta\Pi \approx r\Pi\Delta t.$$

If we substitute equations back we obtain the Black-Scholes differential equation:

Theorem 6.2:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

Boundary conditions

To obtain prices from the Black-Scholes PDE we impose boundary conditions, e.g., for a European call option with strike X and expiring at time T the boundary condition is $f(S,T) = \max\{S - X, 0\}$ for all S. For a European put option with strike X and expiring at time T the boundary condition is $f(S,T) = \max\{X - S, 0\}$ for all S. PDE Theory shows that we obtain a *unique* solution by imposing a further boundary condition at $S = 0$.

If $S_t = 0$ at any time $0 \leq t \leq T$, $S_t = 0$ for all $t \geq T$. So $f(0,t) = e^{-r(T-t)} f(0,T)$ which provides us with a boundary condition. E.g., in the case of European call options, $f(0,t) = 0$. (For numerical purposes we impose boundary conditions at $S = \infty$).

An Example

$f(S,t) = e^{rt} S^{1-2r/\sigma^2}$ is a solution of the Black-Scholes PDE:
$\dfrac{\partial f}{\partial t} = rf(S,t)$, $\dfrac{\partial f}{\partial S} = (1 - \dfrac{2r}{\sigma^2}) e^{rt} S^{-2r/\sigma^2}$

$$\frac{\partial^2 f}{\partial S^2} = -(1 - \frac{2r}{\sigma^2}) \frac{2r}{\sigma^2} e^{rt} S^{-1-2r/\sigma^2}$$

and

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} =$$

$$rf(S,t) + rS(1 - \frac{2r}{\sigma^2}) e^{rt} S^{-2r/\sigma^2} - \frac{1}{2} \sigma^2 S^2 (1 - \frac{2r}{\sigma^2}) \frac{2r}{\sigma^2} e^{rt} S^{-1-2r/\sigma^2} =$$

$$rf(S,t) + r(1 - \frac{2r}{\sigma^2}) e^{rt} \left[S \times S^{-2r/\sigma^2} - S^2 \times S^{-1-2r/\sigma^2} \right] = rf(S,t) + 0$$

Consider a derivative on certain stock with single payoff at time $T > 0$ amounting to S_T^{1-2r/σ^2} where r is the constant interest rate. Assume that $1 - 2r/\sigma^2 > 0$. Find the value of the derivative at time $0 \le t \le T$.

Consider a portfolio consisting of e^{rT} derivatives and write $v(S, t)$ for the price of this portfolio.

$v(S_T, T) = e^{rT} S_T^{1-2r/\sigma^2} = f(S_T, T)$.

$v(S, t)$ and $f(S, t)$ satisfy the Black-Scholes PDE

$v(0, t) = f(0, t) = 0$, and $v(S, T) = f(S, T)$ hence

$v(S, t) = f(S, t) = e^{rt} S^{1-2r/\sigma^2}$.

So the price of one derivative is

$e^{-rT} e^{rt} S^{1-2r/\sigma^2} = e^{-r(T-t)} S^{1-2r/\sigma^2}$.

The Black-Scholes pricing formulas

Theorem 6.3 (The Black-Scholes pricing formulas:) Consider a European option at time t on stock with spot price S, with strike price X and expiring at time T. Let σ be the annual volatility of the stock, and r the T-year interest rate. Define

$$d_1 = \frac{\log(S/X) + \left(r + \sigma^2/2\right)(T - t)}{\sigma\sqrt{T - t}},$$

$$d_2 = \frac{\log(S/X) + \left(r - \sigma^2/2\right)(T - t)}{\sigma\sqrt{T - t}} = d_1 - \sigma\sqrt{T - t}.$$

Then the price of the call option at time t is
$c = S\Phi(d_1) - Xe^{-r(T-t)}\Phi(d_2)$ and the price of the put option is
$p = Xe^{-r(T-t)}\Phi(-d_2) - S\Phi(-d_1)$.
where Φ is the standard normal distribution function.

Proof: Define the function $c(S,t) = S_t\Phi(d_1) - Xe^{-r(T-t)}\Phi(d_2)$.
In view of the BS differential equation we need to verify that

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} = rc,$$

$$\lim_{t\to T} c(S_t,t) = \max\{S_T - X, 0\}$$

$$\lim_{S\to 0} c(S,t) = 0.$$

The verification of both these statements is straightforward (but tedious!)
A similar argument produces the value of p.

The Black-Scholes pricing formulas are a result of a no-arbitrage argument: if violated use portfolio Π to get a free lunch.
In practice one cannot adjust Π continuously,
and if there are trading charges, one cannot even make frequent adjustments.

The Black-Scholes pricing formulas: the risk neutral valuation approach.

Recall the BS PDE :

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf.$$

There is no μ here!
This formula holds regardless of amount of risk-aversion.
We might as well assume that investors are risk neutral!
Risk-neutral investors only care about the expected return of their investments, and they do not care about uncertainty regarding these returns.
So all investments in a risk neutral world must have the same expected return r equal to the risk-free interest rate.

The principle of risk-neutral valuation

Let f be the price of a derivative which pays $H(S_T)$ for some function H at a future time T.
In our risk-neutral world the stock price has expected return r, the risk-free T-year interest rate.
In a risk-neutral world the current value of the derivative $f(S, 0)$ is the present value of the expected value of the derivative payoff at time T, i.e.,

$$f(S, 0) = e^{-rT}\widetilde{\mathrm{E}}\left(H\left(S_T\right)\right)$$

where $\widetilde{\mathrm{E}}$ denotes expected values in our risk-neutral world.

Example: digital options

Consider a derivative on a stock which at expiration time T pays £1 if $S_T \leq a$, for some positive number a, and zero otherwise. (These options are known as *digital* or *binary* options.)
Let the volatility of the stock price be σ and assume all interest rates are constant and equal to r.
Apply a risk neutral valuation argument to show that, for any $0 \leq t \leq T$, the value of this derivative equals

$$e^{-r(T-t)}\Phi\left(\frac{\log\left(a/S\right)-\left(r-\sigma^2/2\right)\left(T-t\right)}{\sigma\sqrt{T-t}}\right)$$

where Φ is the cumulative distribution function of the standard normal distribution.

Solution

We are assuming S follows the process

$$dS = \mu S dt + \sigma S dB$$

for constants μ and σ, and so at time $0 \leq t \leq T$, $\log S_T$ is normally distributed with mean $\log S + (\mu - \frac{\sigma^2}{2})(T - t)$ and standard deviation $\sigma\sqrt{T - t}$.
In a risk neutral world we set $\mu = r$
and now $\log S_T$ is normally distributed with mean $\log S + (r - \frac{\sigma^2}{2})(T - t)$ and standard deviation $\sigma\sqrt{T - t}$.

The event $S_T \leq a$ is equivalent to the event $\log S_T \leq \log a$ and so the probability in this risk neutral world of the event $S_T \leq a$ is $\Phi\left(\frac{\log a - (\log S + (r - \frac{\sigma^2}{2})(T-t))}{\sigma\sqrt{T-t}}\right) = \Phi\left(\frac{\log a/S - (r - \frac{\sigma^2}{2})(T-t))}{\sigma\sqrt{(T-t)}}\right)$. In our risk neutral world the value of the derivative is the present value of the expected value of its payoff, i.e.,

$$e^{-r(T-t)}\Phi\left(\frac{\log(a/S) - \left(r - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

Back to European call/put options

Again, $\log S_T$ is normally distributed with mean $\log S + \left(\mu - \frac{\sigma^2}{2}\right) T$ and variance $\sigma^2 T$; in our risk neutral valuation argument we set $\mu = r$.
Risk-neutral investors also expect the current value of the derivative $f(S, 0)$ to be the expected value of the present value of the derivative payoff at time T, i.e.,

$$f(S, 0) = e^{-rT} \widetilde{E}\left(H\left(S_T\right)\right).$$

Consider the case where $H(y) = \max\{y - X, 0\}$ with X being the strike price of the call option. We have
$c = e^{-rT} \widetilde{E}\left(\max\{S_T - X, 0\}\right)$.

Let ϕ be the density function of the lognormal random variable S_T in our risk neutral world.

$$c = e^{-rT} \int_X^\infty (y - X)\phi(y)\, dy.$$

Lemma:

$$\int_X^\infty (y - X)\phi(y)\, dy = Se^{rT}\Phi(d_1) - X\Phi(d_2)$$

where $d_1 = \frac{\log(S/X) + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}}$,
$d_2 = \frac{\log(S/X) + \left(r - \sigma^2/2\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$.

Now we obtain

$$c = S\Phi\left(d_1\right) - e^{-rT} X\Phi\left(d_2\right)$$

and a similar argument shows that the price of a European put option with strike price X is

$$p = Xe^{-rT}\Phi(-d_2) - S\Phi(-d_1)$$

Volatility

The parameters European options are the spot price of the stock, the strike price, time to expiration, the interest rate, and the volatility σ of the stock price. The first four parameters are always known; but volatility is not directly observable. One can estimate *historical volatility* by analysing the time series consisting of prices of the stock at previous times in the past. There are many problems involved with these estimates. One such problem is that even though our model for stock prices assumes constant volatility, in practice some periods of time, e.g., immediately after September 11th, 2001, are more volatile than others. So we might want to give lower weights to more distant measurements.

Traders do not normally use historical volatility when applying the Black-Scholes pricing formulas.
Instead they use implied volatilities which are the value of the volatility parameter which will produce the observed market price of a given option.
This sounds like a circular argument, but it is useful for example to produce prices of an option based on a similar one and to produce new prices as the price of the underlying stock changes or as time progresses.

The end