Portfolio Theory

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Portfolio Theory

$\boxed{\text{Aim: To find optimal investment strategies.}}$

Investors' preferences:

How does a person decide which investment is best?
Is "Best" = "highest expected return"?

Consider the following £1,000 one-year investments:
Portfolio A: Will be worth £1,100 with probability 1.
Portfolio B: Will be worth £1,000 with probability 1/2 and £1,300 with probability 1/2.
Portfolio C: Will be worth £500 with probability 1/10, £1,200 with probability 8/10 and £3,000 with probability 1/10.
The expected returns are

$$r_A = \frac{1100 - 1000}{1000} = 0.1,$$

$$r_B = \frac{1}{2}\frac{1000 - 1000}{1000} + \frac{1}{2}\frac{1300 - 1000}{1000} = 0.15$$

$$r_C = \frac{1}{10}\frac{500 - 1000}{1000} + \frac{8}{10}\frac{1200 - 1000}{1000} + \frac{1}{10}\frac{3000 - 1000}{1000}$$
$$= 0.31$$

Now consider the following investors:
Mr. X: Wants to sail around the world on a cruise costing £1,200 a year from now.
Ms. Y: Must repay her mortgage in a year and must have £1,100 to do so.
Dr. Z: Needs to buy a rare book worth £1,300.
The chances of Mr X. sailing around the world if he invests in investments A,B or C are 0, $1/2$ and $9/10$ respectively, so he should be advised to invest in C.
Only investment A guarantees a return sufficient for Ms. Y to pay her mortgage and she should choose it.
The probability of the investments being worth at least £1,300 after a year are 0, $1/2$ and $1/10$ respectively, so Dr. Z should invest in portfolio B.

So different investors prefer different investments!
"highest expected returns" $\neq$ "optimal": the *whole distribution* of the returns needs to be taken into account.
Instead of considering the whole distribution of the returns of an investment we take into account two parameters:
the expected return which we denote r and the
standard deviation of the return which we denote σ.
We now rephrase our problem: given a set of investments whose returns have known expected values and standard deviations, which one is "optimal"?

Axioms satisfied by preferences

Consider two investments A and B with expected returns r_A and r_B and standard deviation of returns σ_A and σ_B.

The following assumptions look plausible:

A1 *Investors are greedy:*

If $\sigma_A = \sigma_B$ and $r_A > r_B$ investors prefer A to B.

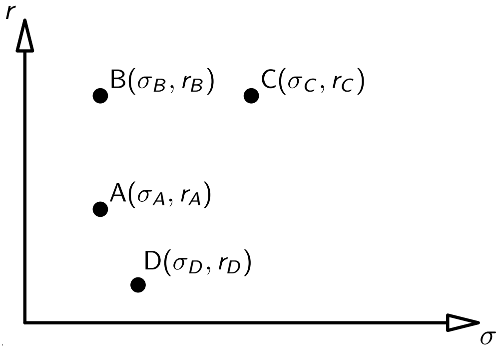
A2 *Investors are risk averse:*

If $r_A = r_B$ and $\sigma_A > \sigma_B$ investors prefer B to A.

A3 *Transitivity of preferences:*

If investment B is preferable to A and if investment C is preferable to B then investment C is preferable to A.

We are introducing a partial ordering $\prec$ on the points of the σ-r plane:

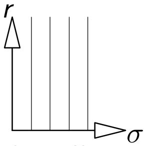


B is preferable to both A and C. Investments A and C are incomparable.

Indifference curves

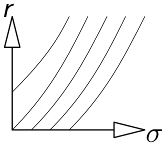
We describe the preferences of an investor by specifying the sets of investments which are equally attractive to the given investor.
We define an indifference curve of an investor: this is the curve consisting of points (σ, r) for which investments with these expected returns and standard deviation of returns are all equally attractive to our investor.
Notice that assumptions A1, A2 and A3 imply that these curves must be non-decreasing. (Why?)

Consider hypothetical investors X,Y,Z and W with the following indifference curves.



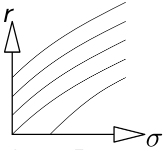
Investor X

Investor X cannot tolerate uncertainty at all: this investor prefers a certain zero return rather a very large expected return with a small degree of uncertainty.



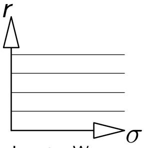
Investor Y

Investor Y is willing to take some risks



Investor Z

but not as much as investor Z
(his indifference curves are steeper.)



Investor W

Investor W is risk neutral.

Portfolios consisting entirely of risky investments

Consider two investments A and B with expected returns r_A and r_B and standard deviation of returns σ_A and σ_B.
We split an investment of £1 between the two investments: consider portfolio Π_t consisting of t units of investment A and $1 - t$ units of investment B.

We can do this for *any* t and not just $0 \leq t \leq 1$.
For example, to construct portfolio Π_2 we short sell £1 worth of B and buy £2 worth of A, for a total investment of £1.

Let A and B be the random variables representing the annual return of investments A and B.
The variance of Π_t is given by $Var(\Pi_t) = Var(tA + (1-t)B)$
$= Var(tA) + Var((1-t)B) + 2Covar(tA, (1-t)B)$
$= t^2 Var(A) + (1-t)^2 Var(B) + 2t(1-t)Covar(A, B)$
$= t^2 Var(A) + (1-t)^2 Var(B) + 2t(1-t)\rho(A, B)\sqrt{Var(A)Var(B)}$
$= (t\sigma_A)^2 + 2t(1-t)\rho(A, B)\sigma_A\sigma_B + ((1-t)\sigma_B)^2$

The shapes of these curves are <u>concave</u>:

Proposition 7.1: The curve in the σ-r plane given parametrically by $\left(\sqrt{(t\sigma_A)^2 + 2t(1-t)\rho(A,B)\sigma_A\sigma_B + ((1-t)\sigma_B)^2},\right.$ $\left. tr_A + (1-t)r_B\right)$ for $0 \le t \le 1$ lies to the left of the line segment connecting the points (σ_A, r_A) and (σ_B, r_B).

Proof: Since $\rho(A,B) \le 1$,

$$\sqrt{(t\sigma_A)^2 + 2t(1-t)\rho(A,B)\sigma_A\sigma_B + ((1-t)\sigma_B)^2}$$
$$\le \sqrt{(t\sigma_A)^2 + 2t(1-t)\sigma_A\sigma_B + ((1-t)\sigma_B)^2}$$
$$= \sqrt{(t\sigma_A + (1-t)\sigma_B)^2} = t\sigma_A + (1-t)\sigma_B.$$

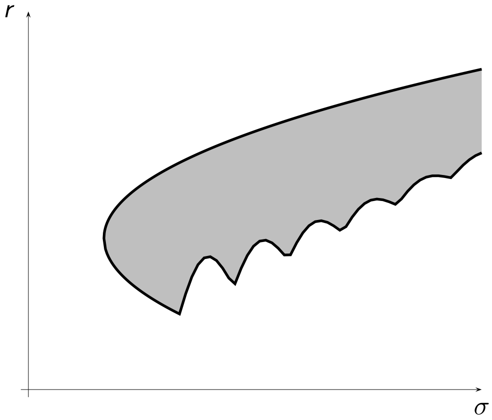
The result follows from the fact that the parametric equation of the line segment connecting the points (σ_A, r_A) and (σ_B, r_B) is

$$\left\{(t\sigma_A + (1-t)\sigma_B, tr_A + (1-t)r_B) \mid 0 \le t \le 1\right\}.$$

The feasible set

Suppose now that there are many different investments $A_1, \ldots, A_n$ available.
We can invest our one unit of currency by investing t_i in A_i for each $1 \leq i \leq n$ as long as $\sum_{i=1}^{n} t_i = 1$.
What are all possible pairs (σ, r) corresponding to these portfolios?
This set of points in the σ-r plane is called the feasible set.

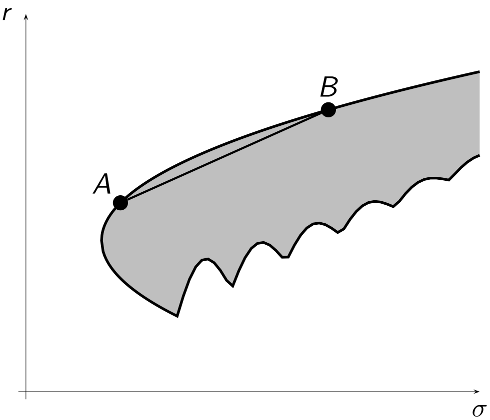
Feasible sets



Efficient portfolios

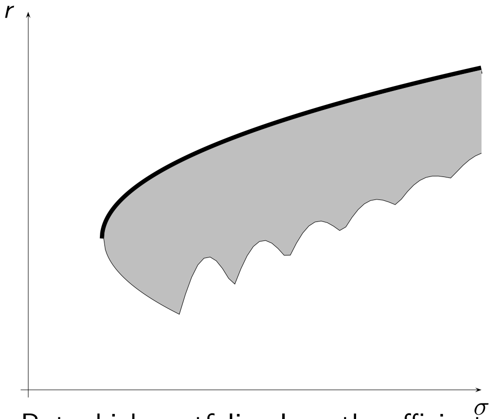
Which portfolios among all possible ones should an investor satisfying axioms A1,A2 and A3 choose?
Definition:
An efficient portfolio is a feasible portfolio that provides the greatest expected return for a given level of risk, or equivalently, the lowest risk for a given expected return. This is also called an optimal portfolio. The efficient frontier is the set of all efficient portfolios. Obviously, our investor should choose a portfolio along the efficient frontier!

Feasible sets are convex along efficient frontier:



The feasible set is *convex* along the efficient frontier, in the sense that for any two portfolios A and B in the feasible set, there exist feasible portfolios above the portfolios in the segment connecting A and B.

Which efficient portfolio do we choose?

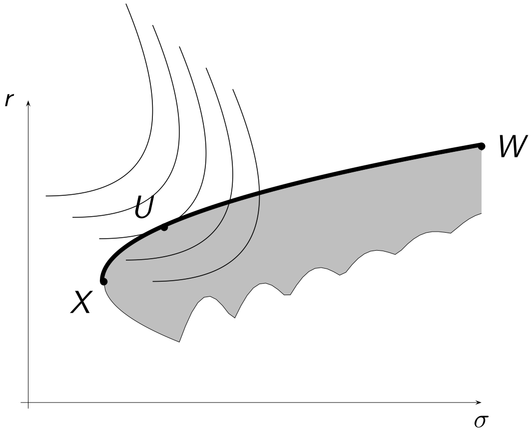


But which portfolio along the efficient frontier will our investor choose?

This is where risk preferences start playing a role.

Different choices of portfolios for different appetites for risk

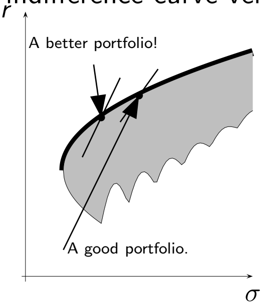
Consider investors X (with no risk tolerance at all) and W (risk neutral) discussed before together with investor U whose indifference curves are given below.



Optimal portfolios occur where indifference curves are tangent to the efficient frontier.

If the indifference curves are not too badly behaved, e.g., if the indifference curves are the level curves of some smooth function $F(\sigma, r)$, then we should expect the optimal portfolio to be at a point where the indifference curve is tangent to the efficient frontier.

Otherwise, if it occurs at a point where the indifference curve intersects the efficient frontier transversally, find an almost parallel indifference curve very close to the original one and to its left.



Portfolios on this curve are more desirable and, if we chose the second indifference curve close enough to the original one, it will also intersect the efficient frontier, and this intersection will correspond to a better choice of portfolio than the one corresponding to the original point of intersection.

Portfolios containing risk-free investments

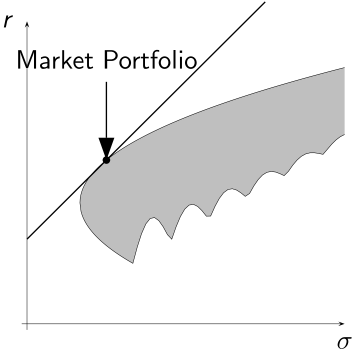
We now add a risk-free investment B. Let r_B be its (expected) return. Since r_B is constant, its covariance with the returns of any other portfolio Π is zero so the portfolio Π_t consisting of t units of currency invested in B and $(1-t)$ units of currency invested in Π has expected return

$$E\left(tB + (1-t)\Pi\right) = tr_B + (1-t)r_\Pi$$

(where we used B and Π to denote also the returns of the investments B and Π) and standard deviation of return
$\sqrt{Var(tB + (1-t)\Pi)} = \sqrt{Var((1-t)\Pi)}$
$= \sqrt{(1-t)^2 Var(\Pi)} = |1-t|\sigma_\Pi.$

The curve $t \mapsto (\sigma_{\Pi_t}, r_{\Pi_t})$ for $t \leq 1$ is a straight line passing through the points $(0, r_B)$ and (σ_Π, r_Π) and all the points on or below such a line will be part of the feasible set.
What happens to the efficient frontier? Consider the set S consisting of all the slopes s of lines ℓ_s in the σ-r plane which pass through the point $(0, r_B)$ and intersect the feasible set. Let m be the supremum of S. Consider now the line ℓ_m which is above all the others:

The line ℓ_m will either be tangent to the efficient frontier or asymptotic to it.
We will see in Chapter 8 that, if we impose additional conditions on markets and investors, ℓ_m cannot be an asymptote of the efficient frontier and so it is tangent to it.



This point of tangency is called the *market portfolio* and we shall denote the corresponding portfolio with M.
The new efficient frontier, ℓ_m is called the *capital market line*.

We just proved the following:
Theorem 7.2:
In the presence of a risk-free investment there exists an (essentially) unique investment choice consisting entirely of risky investments which is efficient, namely, the market portfolio.
Any other efficient investment is a combination of an investment in the market portfolio and in the risk-free investment.

The End