# The Capital Asset Pricing Model

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Aim: To find the "correct" price of financial assets.

**Additional assumptions about markets and investors:**

**A4:** Markets are in equilibrium: The total demand for any financial instrument equals its total supply.

**A5:** Uniform horizon: All investors are investing for the same period of time.

**A6:** Homogeneity: All investors agree on the expected returns of investments, their standard deviations of returns and the correlations between these returns. All investors can borrow and lend unlimited amounts of money at the same uniform risk-free rate.

**A7:** No friction: There are no transaction costs and no taxes.

# The market portfolio.

The market portfolio is the only efficient portfolio consisting entirely of risky investments.
What is this portfolio?
**Theorem 8.1:**
Let $I_1, I_2, \ldots, I_n$ be *all* risky investments,
and assume their total market value is $w_1, w_2, \ldots, w_n$.
Let $W = w_1 + w_2 + \cdots + w_n$.
The market portfolio consists of a portfolio of investments of $\frac{w_j}{W}$ in $I_j$ for each $1 \leq j \leq n$.

**Proof:** Assume that there are $m$ investors and that for any $1 \leq k \leq m$ investor $k$ holds $h_{kj}$ worth of investment $j$.
Write $H_k = \sum_{j=1}^{n} h_{kj}$ for the total amount invested in risky investments for investor $k$.
Axioms A1, A2, A3, the results of Chapter 7 and Axioms A5, A6 imply that all investors will have the same proportion of each risky investment, i.e., $\dfrac{h_{kj}}{H_k} = \dfrac{h_{lj}}{H_l}$ for every $1 \leq k, l \leq m$.

Axiom A4 implies that for each $1 \leq j \leq n$,
$w_j = \sum_{k=1}^{m} h_{kj}$. (Notice $w_j$ is the total supply of investment $j$ while $\sum_{k=1}^{m} h_{kj}$ is its total demand; the two should be equal if the market is in equilibrium.) Summing over all investments gives
$W = \sum_{k=1}^{m} H_k$.
Now the proportion of the market portfolio invested in $I_j$ is the same as any investor's proportion of investment in $I_j$ out of the total risky investments, so we can write this proportion as $\frac{h_{1j}}{H_1}$

$$= \frac{h_{1j} \sum_{k=1}^{m} H_k}{H_1 \sum_{k=1}^{m} H_k} = \frac{h_{1j} \left(1 + \sum_{k=2}^{m} \frac{H_k}{H_1}\right)}{\sum_{k=1}^{m} H_k} = \frac{h_{1j} + \sum_{k=2}^{m} h_{kj}}{W}$$

$$= \frac{\sum_{k=1}^{m} h_{kj}}{W} = \frac{w_j}{W}$$

# Stock market indices

You may have heard of stock market indices (e.g., FTS100, the Dow Jones Industrial Index, S&P 500, Nikkei, CAC40, DAX, etc.)
The values of these indices are the weighted average price of a set of stocks, where the weights are proportional to the proportion of the total value of a stock as part of the total value of the whole set of stocks.
You might have also heard of tracker funds: these are investments that hold shares in the same proportion as a given index, e.g., FTSE 100 trackers.
The previous theorem says roughly that the only risky investments in the portfolio of an investor who assumes axioms A1-A7 must be tracker funds.

# Actively managed funds

There are other types of investment funds, actively managed funds. These funds invest money in stocks carefully chosen by spectacularly highly paid fund managers.

These funds demand high fees from investors in return for applying their talents in choosing the way in which the fund's assets will be invested.

So you are asked to pay large fees to have axiom A6 broken: these fund managers claim to possess knowledge which is not apparent to lesser investors.

There is a ongoing debate on whether these fund managers are worth these high fees.

# The market price of risk

For any efficient investment A lying on the market line we have

$$r_A - r_B = \frac{r_M - r_B}{\sigma_M}\sigma_A$$

where $r_B$ is the risk-free interest rate and $M$ is the market portfolio.
We interpret $\frac{r_M - r_B}{\sigma_M}$ as the market price of risk:
this slope measures how much more return investors demand for an increase of one unit in the volatility of their returns.
We want a similar expression for the excess return above the risk-free interest rate for non-efficient portfolios, e.g., individual stocks.

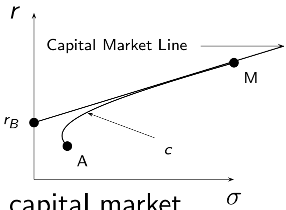
**Theorem 8.2:**

For any portfolio A we have

$$r_A - r_B = \frac{r_M - r_B}{\sigma_M^2}\,\mathrm{Cov}(A, M)$$

where $r_B$ is the risk-free interest rate, $M$ is the market portfolio and $\mathrm{Cov}(A, M)$ is the covariance between the return of A and the return of M.

For any $0 \leq t \leq 1$, let portfolio $\Pi_t$ consist of an investment of $t$ in A and an investment of $1 - t$ in M.




The curve $c$ given by $(\sigma_{\Pi_t}, r_{\Pi_t})$ in the $\sigma$-$r$ plane joins points $A$ and $M$.

$c$ intersects the capital market line at $M$, and the capital market line must be tangent to $c$ at $M$:

Otherwise $c$ would *cross* the capital market line and we would have a portfolio above the capital market line, contradicting the fact that the capital market line is the efficient frontier.

Calculate the slope of $c$ at $M$: $c$ is given by $t \mapsto \left(\sqrt{t^2\sigma_A^2 + 2t(1-t)\operatorname{Cov}(A,M) + (1-t)^2\sigma_M^2},\ tr_A + (1-t)r_M\right)$ $(0 \leq t \leq 1)$.
Evaluate the derivative with respect to $t$

$$\left(\frac{2t\sigma_A^2 + (2-4t)\operatorname{Cov}(A,M) - 2(1-t)\sigma_M^2}{2\sqrt{t^2\sigma_A^2 + 2t(1-t)\operatorname{Cov}(A,M) + (1-t)^2\sigma_M^2}}, r_A - r_M\right)$$

at $t = 0$ to obtain the slope at point $M$: $\frac{r_A - r_M}{\operatorname{Cov}(A,M) - \sigma_M^2}\sigma_M$. But this slope must be equal to the slope of the capital market line, i.e., $\dfrac{r_A - r_M}{\operatorname{Cov}(A,M) - \sigma_M^2}\sigma_M = \dfrac{r_M - r_B}{\sigma_M}$ and we can rearrange this to obtain $r_A - r_B = \dfrac{r_M - r_B}{\sigma_M^2}\operatorname{Cov}(A,M)$.

# The beta coefficient

**Definition:** The *beta coefficient* of a portfolio A is defined as

$$\beta = \frac{\mathrm{Cov}(A, M)}{\sigma_M^2}.$$

The *security market line* is the linear relation between expected returns $r$ and beta coefficients $\beta$ given by

$$r = r_B + (r_M - r_B)\beta.$$

We can restate the previous theorem:
*for any investment with expected return $r$ and beta coefficient $\beta$ the point $(\beta, r)$ lies on the security market line.*

# An Example

You are given the following data on three stocks and the market portfolio:

| | Expected return | Correlation with market portfolio | Standard deviation of return |
|---|---|---|---|
| Stock 1 | **?** | 0.8 | 20% |
| Stock 2 | 16% | **?** | 18% |
| Stock 3 | 17% | 0.6 | **?** |
| Market portfolio | 15% | 1 | 12% |

The risk-free interest rate for the period is 10%.

Give the equation of the capital market line and fill in all missing data in the table above.

The capital market line is given by $r - 0.1 = \frac{0.15 - 0.1}{0.12}\sigma$
$\Rightarrow r = \frac{5}{12}\sigma + \frac{1}{10}$. The beta coefficient of Stock 1 is

$$\beta_1 = \frac{\text{Cov}(\text{Stock 1}, M)}{\sigma_M^2}$$

$$= \frac{\rho(\text{Stock 1}, M)\sigma_1}{\sigma_M} = \frac{0.8 \times 0.2}{0.12} = \frac{4}{3}.$$

We obtain the expected return using the security market line
$r_1 = r_B + (r_M - r_B)\beta_1 = 0.1 + (0.15 - 0.1) \times \frac{4}{3} = \frac{1}{6}$.

The beta coefficient $\beta_2$ of Stock 2 must satisfy

$$r_2 = r_B + (r_M - r_B)\beta_2$$

$$\Rightarrow \beta_2 = \frac{0.16 - 0.1}{0.15 - 0.10} = \frac{6}{5}$$

and so the correlation with the market portfolio is

$$\rho(\text{Stock } 2, M) = \frac{\text{Cov}(\text{Stock } 2, M)}{\sigma_2 \sigma_M}$$

$$= \frac{\beta_2 \sigma_M}{\sigma_2} = \frac{6}{5}\frac{\times 0.12}{0.18} = \frac{4}{5}.$$

The beta coefficient $\beta_3$ of Stock 3 must also satisfy $r_3 = r_B + (r_M - r_B)\beta_3 \Rightarrow \beta_3 = \dfrac{0.17 - 0.1}{0.15 - 0.1} = \dfrac{7}{5}$ but $\beta_3 = \dfrac{\text{Cov}(\text{Stock } 3, M)}{\sigma_M^2} = \dfrac{\rho(\text{Stock } 3, M)\sigma_3}{\sigma_M}$ so $\sigma_3 = \dfrac{\sigma_M \beta_3}{\rho(\text{Stock } 3, M)} = \dfrac{7}{5}\dfrac{0.12}{0.6} = 28\%$.

# Another example

The year is 3003 and there are only three companies left in the universe: Exxon with 5 billion outstanding shares, Microsoft with 4 billion outstanding shares and Monsanto with 2 billion outstanding shares.
The shares trade at 3,5 and 6 USD, respectively.
The market portfolio consists of investments in these three companies in the proportions

$$\frac{5\times 10^9\times 3}{5\times 10^9\times 3+4\times 10^9\times 5+2\times 10^9\times 6}=\frac{15}{47},$$
$$\frac{4\times 10^9\times 5}{5\times 10^9\times 3+4\times 10^9\times 5+2\times 10^9\times 6}=\frac{20}{47},$$
$$\frac{2\times 10^9\times 6}{5\times 10^9\times 3+4\times 10^9\times 5+2\times 10^9\times 6}=\frac{12}{47}.$$

Assume that one year into the future the universe will be in one of three states each one occurring with probability 1/3.
The share prices in each of these states are given by:

| | State 1 | State 2 | State 3 |
|---|---|---|---|
| Exxon | 1 | 4 | 8 |
| Microsoft | 4 | 6 | 7 |
| Monsanto | 1 | 7 | 24 |

We now proceed to find the beta coefficients of each of the three stocks.

The returns of the stocks and the market portfolio in each of the three states are given by:

| | State 1 | State 2 | State 3 |
|---|---|---|---|
| Exxon | -2/3 | 1/3 | 5/3 |
| Microsoft | -1/5 | 1/5 | 2/5 |
| Monsanto | -5/6 | 1/6 | 3 |
| Market Portfolio | -24/47 | 11/47 | 69/47 |

and so the expected returns are $\frac{1}{3} \times \frac{-2}{3} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{5}{3} = \frac{4}{9}$, $\frac{1}{3} \times \frac{-1}{5} + \frac{1}{3} \times \frac{1}{5} + \frac{1}{3}\frac{2}{5} = \frac{2}{15}$, $\frac{1}{3} \times \frac{-5}{6} + \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times 3 = \frac{7}{9}$ and $\frac{1}{3} \times \frac{-24}{47} + \frac{1}{3} \times \frac{11}{47} + \frac{1}{3} \times \frac{69}{47} = \frac{56}{141}$.

We now find $\sigma_M^2$:

$$\sigma_M^2 =$$

$$\left(\frac{1}{3} \times \left(\frac{-24}{47}\right)^2 + \frac{1}{3} \times \left(\frac{11}{47}\right)^2 + \frac{1}{3} \times \left(\frac{69}{47}\right)^2\right) - \left(\frac{56}{141}\right)^2$$

$$\approx 0.66586$$

The covariance between the return of Exxon shares and the market portfolio is given by

$$\frac{1}{3} \times \frac{-2}{3} \times \frac{-24}{47} + \frac{1}{3} \times \frac{1}{3} \times \frac{11}{47} + \frac{1}{3} \times \frac{5}{3} \times \frac{69}{47} - \frac{4}{9} \times \frac{56}{141} \approx 0.77857.$$

The corresponding beta coefficient is $\approx 1.1693$.

The covariance between the return of Microsoft shares and the market portfolio is given by

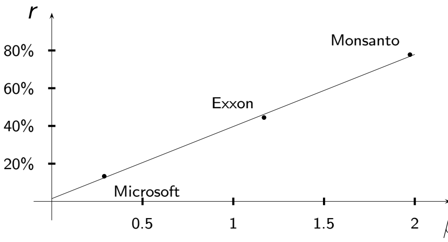
$$\frac{1}{3} \times \frac{-1}{5} \times \frac{-24}{47} + \frac{1}{3} \times \frac{1}{5} \times \frac{11}{47} + \frac{1}{3} \times \frac{2}{5} \times \frac{69}{47} - \frac{2}{15} \times \frac{56}{141} \approx 0.19243.$$

The corresponding beta coefficient is $\approx 0.289$.
The covariance between the return of Monsanto shares and the market portfolio is given by

$$\frac{1}{3} \times \frac{-5}{6} \times \frac{-24}{47} + \frac{1}{3} \times \frac{1}{6} \times \frac{11}{47} + \frac{1}{3} \times 3 \times \frac{69}{47} - \frac{7}{9} \times \frac{56}{141} \approx 1.314.$$

The corresponding beta coefficient is $\approx 1.9734$.

We are told that the one-year risk-free interest rate $r_B$ is 3%. We plot the security market line in the $\beta$-$r$ plane.




The points in the $\beta$-$r$ plane corresponding to the three companies' stock lie very close to the security market,

but they do not lie exactly on it.

Can an investor take advantage of these discrepancies?

Not necessarily, since the result in Theorem 8.2 is not based on no-arbitrage arguments, and so there is no apparent way to translate discrepancies between real-life data and CAPM predictions into arbitrage strategies.

# Market risk and unique risk

Consider the random variables $A$ and $M$ representing the returns of a portfolio ("portfolio A") and the market portfolio.
If there were a linear relationship between these two, the security market line $r = r_B + (r_M - r_B)\beta$ would suggest that $A = r_B + (M - r_B)\beta$ and in which case we would have $\sigma_A = \beta\sigma_M$.
In practice there is no linear relation between $A$ and $M$, but we can always write $A = \beta M + (A - \beta M)$ and we (rather arbitrarily) assume that $\beta M$ and $(A - \beta M)$ have zero correlation.

We are trying to separate the risk of $A$ into a risk originating from the market plus a risk specific to $A$.
We refer to $\beta\sigma_M$ as the *market risk* of A
and to $\sqrt{Var(A-\beta M)} = \sqrt{\sigma_A^2 + \beta^2\sigma_M^2 - 2\,\mathrm{Cov}(A, \beta M)}$
$= \sqrt{\sigma_A^2 + \frac{\mathrm{Cov}(A, M)^2}{\sigma_M^2} - 2\frac{\mathrm{Cov}(A, M)^2}{\sigma_M^2}} = \sqrt{\sigma_A^2 - \beta^2\sigma_M^2}$
as the *unique risk* of A. These two components of risk reflect the fact that the risk of an investment originates partly from fluctuations in the whole market and partly from the uncertainty related to the particular investment.

# The End