MAS362 brief notes

(material shown on slides during the lectures in compact, printer-friendly way.)

This is not a textbook– annotate these with the content of lectures.

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Contents

What is this course about?

In this course you will learn some basic facts about an important component of modern life: *finance*. This is a mathematics course hence we will be interested in *mathematical ideas* used in the context of finance.

Specifically, we answer two questions:

(1) What is the "correct price" of financial assets?
(2) What are optimal investment strategies?

The first question will lead us to the *Black-Scholes pricing formula,* which earned Robert C. Merton and Myron S. Scholes the 1997 Nobel prize in economics. William F. Sharpe and Harry M. Markowitz received the 1990 Nobel prize in economics for answering the second question.

1. What does "correct price" mean?

Lets first see an example of an *incorrect price*. Suppose that a barrel of oil trades at £100 (i.e., one can buy it for £100 and one can sell it for £100). I want to start trading in oil, too. Would £120 be a correct price? No! Other traders will buy oil in large quantities for £100, sell it to me for £120. Everyone except me will become rich, I end up with a pile of barrels of oil I can't sell. Would £90 be a correct price?

A more subtle example Consider three merchants who are willing to buy and sell bags containing apples and oranges (all of identical size and quality) as follows:

	Bag content	Price
Merchant I	3 apples, 2 oranges	£5
Merchant II	2 apples, 3 oranges	£6
Merchant III	4 apples, 3 oranges	£8

To see that these prices are incorrect, let me show you how to get rich:

(a) borrow £36 for a short while (with negligible interest);
(b) buy 6 bags from merchant I and buy 1 bag from merchant II for a total cost of £36;
(c) rearrange the fruit in five bags of 4 apples and 3 oranges each;
(d) sell the five bags to merchant III for £40;
(e) return the £36 loan and pocket a £4 profit.

Repeat this process until you are very rich.

Lots and lots of people would be buying from merchants I and II and selling to merchant III. When this happens the prices of bags I and II will rise and those of bag III will fall, and this process will continue until there are no more easy profits to be made.

Notice:

(a) we didn't assume an intrinsic or objective price of apples and oranges,
(b) correct pricing is an equilibrium price,

(c) we expose incorrect pricing by exhibiting profit-making trading strategies which change prices.

The making of a certain profit with no investment is called *arbitrage*. Correct pricing means setting the unique price which does not introduce arbitrage opportunities. The correct price of bag III would be £36/5 as that is the unique price for which the strategy above (or its opposite) does not produce a profit. We call this method of pricing *no-arbitrage pricing*, which is also known colloquially as *There is no such thing as a free lunch!*

2. Portfolio Theory

Are there investment strategies which are better than others? And what could “better mean” in this context? We are greedy, i.e., we want high returns. But average high returns are risky! Suppose you are offered to make a bet on the outcome of tossing a fair coin: heads– you lose everything you own, including your shirt and are forced to live on the street; tails– you double your wealth. Would you bet? This is a fair game, i.e., the expected gain is zero, but most people would not bet. People are not only greedy– they also have an aversion towards risk. Even if a win triples or quadruples their wealth, most people would not place a bet. If we used a coin which produces tails with 99% probability *some* people might choose to place a bet. Now the expected gain may tempt some, but not others.

There is an interplay between expected returns and risk investors are willing to take. So a “good” way to invest is a strategy that compensates for its risk with an appropriately high expected return.

The last part of the course will make this statement precise: we will see that these optimal investments exist and consist of portfolios [1] of risk-free deposits/bonds and market-index trackers.

[1] *Portfolio*: a collection of assets

CHAPTER 1

Interest, present value and bonds

1. Interest and present value

Money can be lent and thus earn *interest*. Interest rates apply for a given period, and interest is often compounded a fixed number of times in that period. The calculation of interest is based on the following convention.

DEFINITION. An amount A is invested for n years at a yearly interest rate r compounded m times a year, yields at the end of this period $A\left(1+\frac{r}{m}\right)^{mn}$.

The notion of interest is linked to another important notion: *present value*.

DEFINITION. The *present value* of a payment occurring in the future, is the present price of the entitlement to that payment.

Suppose that a deposit of £A yields £B when we withdraw it in t years (with $A < B$). We can now ask how much would we pay *now* to receive £B in t years. The answer is, of course £A! Depositing £A now is equivalent to paying that amount now to receive £B in t years. We say that the *present value* of £B paid in t years is £A.

In this course we want to work with continuously compounded interest rates:

DEFINITION. An amount A is invested for t-years accruing a yearly interest of r compounded continuously, yields at the end of this period £Ae^{rt}.

Unless stated otherwise, we always assume that interest is compounded continuously.

2. Bonds and yield curves

DEFINITION. A *bond* is a contract in which the issuer commits to pay the holder of the bond payments of certain amounts on certain dates.

Typically, a bond would specify

- a *face value* which is an amount paid on the,
- a *maturity date* which is the date when final payments are made, and
- *coupons*: a sequence of payments by the issuer: their amounts and payment dates.

Bonds are issued mainly by governments and corporations often in auctions, and can be traded in exchanges (sometimes referred to as secondary markets). [1]

DEFINITION. A *zero coupon bond* has no coupons, i.e., the issuer makes only one payment: the face value is paid at the maturity of the bond.

[1] You can find descriptions of bonds issued by the UK government here and here.

The price P of a £1 face value zero coupon bond with maturity in t years is precisely the amount of money you are willing to pay now in order to be receive £1 in t years. So P is the present value of £1 paid t years into the future.

PROPOSITION 1. *Let P be the price of a zero-coupon bond with face value £1 and maturing in t years. Let r be the (continuously compounded) interest rate for t-year deposits and loans. Then $P = e^{-rt}$.*

In proving this statement we will make our usual assumption about the world: there are no arbitrage opportunities. We also assume that everyone can issue bonds, lend and borrow money under the same terms (these assumptions certainly don't apply to you and I but the do apply approximately to stable governments and large corporations.)

PROOF. If $P > e^{-rt}$,

- issue a zero-coupon bond with face value of £1 and maturing in t years, and collect its price £P,
- deposit £P for t years earning a (continuously compounded) interest rate of r,
- wait t years,
- collect the balance of your deposit amounting to Pe^{rt},
- pay the face value of the bond you issued amounting to £1,
- pocket $Pe^{rt} - 1 > 0$.

This strategy, which required no initial investment and produced a certain profit, is an arbitrage strategy, which we assume not to exist.

If $P < e^{-rt}$,

- borrow £P for t years earning a (continuously compounded) interest rate of r,
- buy a zero-coupon bond with face value of £1 and maturing in t years, and collect its price £P,
- wait t years,
- receive the face value of your bond amounting to £1,
- repay the balance of your loan amounting to Pe^{rt},
- pocket $1 - Pe^{rt} > 0$.

This is another arbitrage strategy, which we assume not to exist.

Since both inequalities cannot occur, we are forced to deduce that $P = e^{-rt}$. □

DEFINITION (Discount curves and yield curves). A *discount curve* is a function of time $P(t)$ whose value at any $t \geq 0$ is the present value of one unit of currency paid in t years. Equivalently, $P(t)$ is the price of a zero coupon bond with face value 1 maturing in t years.

A *yield curve* is a function $Y(t)$ whose value at any $t > 0$ is the interest rate paid on t-year deposits.

The values of $Y(t)$ are also called *spot (interest) rates* or *yields.* [2]

We can now rephrase the previous proposition as $P(t) = e^{-Y(t)t}$.

Yield curves are not directly observable– these are constructed by finding interest rates which fit observed prices of assets whose prices depend on interest rates (e.g., bonds) and obtaining $Y(t)$ from these interest rates by linear interpolation. This process is the *bootstrap method* illustrated in class.

[2] You can check what is the current UK yield curve here.

So far we valued bonds under the assumption that their issuers will make the payments they contracted to make. In real life this is not always the case: e.g., banks go bankrupt [3] and governments default on their debt. [4] The price of bonds and the rates of deposit reflect this: risky bonds are *cheaper* and deposits in shaky banks pay *higher* interest. The same is true of secured loans, e.g., mortgages, compared to non-secured loans.

Henceforth when working with interest rates we will disregard risk. We always assume that for every currency there exists a risk-free institution issuing bonds in that currency and our yield curves and discount factors will refer to these bonds. We also assume that everyone can both deposit and borrow *any* amount of cash for *any* maturity date at these risk-free rates. (Don't try to tell that to your bank manager.)

3. Constructing yield curves: the bootstrap method

We now describe a method of producing yield curves based on the prices of bonds illustrated in the following example.

Consider the following bonds with semi-annual coupons:

Face value (£)	Maturity (years)	Annual interest	Price
100	0.25	0	98.3
100	0.5	0	96.5
100	1	0	93.7
100	1.5	4	95.5
100	2	6	97.2

We can translate the first three lines in the table to

$$P(0.25) = 0.983, Y(0.25) = -\frac{\log 0.983}{0.25} \approx 6.86\%$$
$$P(0.5) = 0.965, Y(0.5) \approx 7.13\%$$
$$P(1) = 0.937, Y(1) \approx 6.51\%.$$

The fourth line implies that

$$2P(0.5) + 2P(1) + 102P(1.5) = 95.5$$

and solving for $P(1.5)$ we obtain $P(1.5) \approx 0.89898$ and $Y(1.5) \approx 7.10\%$.

The fifth line implies that $3P(0.5) + 3P(1) + 3P(1.5) + 103P(2) = 97.2$ and solving for $P(2)$ we obtain $P(2) \approx .8621073672$ and $Y(2) \approx 7.42\%$.

4. Forward rates

Definition. A *forward rate agreement* is a contract in which one party agrees to pay the other party a pre-specified interest rate on a deposit occurring during a specified period of time in the future.

Definition. Let $0 \le t_1 < t_2$ and let r_1 and r_2 be the t_1 and t_2-year interest rates, respectively. The *forward rate* for the period from t_1 to t_2 is $\dfrac{r_2 t_2 - r_1 t_1}{t_2 - t_1}$.

Proposition 2. *In a market with no arbitrage opportunities the interest rates of forward rate agreements are equal to the corresponding forward rates.*

[3] E.g., the Iceland bank crisis.

[4] (E.g., Russian debt default)

PROOF. Let the forward rate agreement time period start in t_1 years and end in t_2 years, let r be the pre-specified interest rate and let

$$r_{12} = \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1}$$

be the corresponding forward rate for the period from t_1 to t_2.

If $r > r_{12}$,

- enter the agreement as a depositor,
- borrow $e^{-r_1 t_1}$ for t_2 years and deposit this amount for t_1 years.
- At time $t = t_1$ receive the balance of your deposit which will be $e^{-r_1 t_1} e^{r_1 t_1} = 1$, and deposit it until $t = t_2$ earning an interest rate of r.
- At time $t = t_2$ obtain the balance of your deposit which will equal $e^{r(t_2 - t_1)}$ and repay the balance of your loan, which will be $e^{-r_1 t_1} e^{r_2 t_2} = e^{r_2 t_2 - r_1 t_1}$.
- Since
$$r > \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1} \Rightarrow e^{r(t_2 - t_1)} > e^{r_2 t_2 - r_1 t_1}$$
you can now pocket the difference $e^{r(t_2 - t_1)} - e^{r_2 t_2 - r_1 t_1} > 0$.

If $r < r_{12}$ adopt the opposite strategy:

- enter the agreement as a borrower,
- borrow $e^{-r_1 t_1}$ for t_1 years and deposit this amount for t_2 years.
- At time $t = t_1$ borrow £1 until $t = t_2$ paying an interest rate of r, and use this £1 to repay your loan.
- At time $t = t_2$ receive the balance of your deposit which will be $e^{-r_1 t_1} e^{r_2 t_2} = e^{r_2 t_2 - r_1 t_1}$, and use it to repay the balance of your second loan which will be $e^{r(t_2 - t_1)}$.
- Now
$$r < \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1} \Rightarrow e^{r(t_2 - t_1)} < e^{r_2 t_2 - r_1 t_1}$$
and you can pocket the difference $e^{r_2 t_2 - r_1 t_1} - e^{r(t_2 - t_1)} > 0$.

□

CHAPTER 2

Forward and Futures Contracts

One of the main aims of this course is to find the correct (i.e., no-arbitrage) prices of derivatives. We now define these formally as follows.

Definition. A *derivative* is an asset whose price depends on the value of other assets.

In this chapter we study two derivatives, namely, *forward contracts* and *futures contracts.* .

Definition. A *forward contract* is an agreement between two parties in which one party agrees to sell a particular asset at an agreed price (the *forward price*), on a certain date (*maturity date* or *delivery date*), and the other party agrees to buy that asset at that price and on that date.

There is an asymmetry in the parties to a forward contract: The party receiving the asset has a *long position* and the party delivering the asset has a *short position.* [1]

Forward contracts are entered freely by both parties and have no cost. The question we now raise is what should the forward price which makes a forward contract valueless be. To answer this question we will distinguish between forward contracts on assets that provide income (e.g., dividend yielding shares in a company) and those that do not (e.g., bars of gold). To do this we'll need to assume short-selling which we describe next

1. Short-selling

Definition. *Short-selling* is selling something which one does not own.

Thus one may own a positive or a negative amount of assets.

In this course we will assume that any tradable asset can be bought, sold and short-sold.

Short-selling sounds vaguely criminal, after all, it involves selling something owned by someone else without their consent– however this is no different from the situation in which a bank takes your deposit and lends it to someone else. As long as the bank can repay your deposit on request, all is well and proper.

How does short-selling work? You short sell an asset through a broker, the broker borrows the asset from another client and sells it, and gives you the proceeds. You can use most of this money to invest in assets through the broker, but you have to keep some of the proceeds in cash held in a *margin account* managed by the broker. The balance on the margin account should be a certain percentage of the spot value of the shot-sold asset, and it is balanced daily: if the spot price of the short-sold asset goes up, the balance in the margin account needs to be increased and you will receive a *margin call*, i.e., a demand to add cash to your margin account. The broker has the right to sell your assets and to use your balance in the margin account to buy back the short-sold asset and return it to the other client, if the other client demands it or if you do not respond to a margin call.

[1] In finance, "short" often means "sell" and "long" often stands for "buy".

To "close" a short position, you buy back asset and return it to its owner. A profit is made if the sale price was higher than the purchase price, i.e., if the asset's price decreased.

We will use repeatedly the following consequence of our ability to short-sell assets:

THEOREM 3. *Let p_1 and p_2 be the current prices of two assets, and suppose that we know with certainty that their prices q_1 and q_2 at some point in the future satisfy $q_1 \leq q_2$. Then $p_1 \leq p_2$. In particular, if we know that two assets will have the same price at some point in the future, then the two assets must have the same price at any prior time.*

2. The forward price

THEOREM 4. *Consider a forward contract on an asset which provides no income and whose price at the present (its* spot *price) is S. The forward price is $F = Se^{rT}$ where T is the time to maturity and $r = Y(T)$, the T-year spot interest rate.*

PROOF. If $F > Se^{rT}$ there is the following arbitrage strategy:

- take a short position in the forward contract,
- borrow £S for T years at spot interest rate of r,
- buy the asset for £S,
- wait T years,
- deliver the asset and collect £F,
- use £Se^{rT} to repay the loan,
- pocket the difference $F - Se^{rT} > 0$.

If $F < Se^{rT}$ there is the following (in a sense opposite) arbitrage strategy:

- take a long position in the forward contract,
- short sell the asset for £S,
- deposit £S for T years at spot interest rate of r,
- wait T years,
- withdraw £Se^{rT} from your deposit,
- have the asset delivered for £F,
- use it to close the short position on the asset,
- pocket the difference $Se^{rT} - F > 0$.

□

We now consider a forward contract on an asset that provides an income during the duration of the forward contract.

THEOREM 5. *Consider a forward contract on an asset that provides income, with maturity date in T years. Let S be the spot price of the asset and let I be the present value of the income generated by the asset until the maturity of the forward contract. Then $F = (S - I)e^{rT}$ where $r = Y(T)$, the T-year spot interest rate.*

PROOF. Suppose that the asset produces payments at times $0 < t_1 < \cdots < t_n < T$ whose present values are $I_1, \ldots, I_n$ and write $r_1 = Y(t_1), \ldots, r_n = Y(t_n)$.

If $F > (S - I)e^{rT}$ there is the following arbitrage strategy:

- Take a short position in the forward contract.
- Borrow £$(S - I)$ for T years at spot interest rate of r.

- for every $1 \leq k \leq n$ borrow I_k for t_k years at a spot rate of r_k. Note that after t_k years the balance of the kth loan will be equal to the kth payment from the asset: I_k is $e^{-r_k t_k} \times k$th payment from the asset.
- Buy the asset for £S.
- Wait T years: meanwhile use the payments from the asset to repay the corresponding loan.
- At the end of this period the outstanding debt is £$(S-I)e^{rT}$.
- Deliver the asset and collect £F.
- Use £$(S-I)e^{rT}$ to repay the loan.
- Pocket the difference $F-(S-I)e^{rT} > 0$.

If $F < (S-I)e^{rT}$ there is the following arbitrage strategy:

- Take a long position in the forward contract.
- Short sell the asset for £S.
- Deposit £$(S-I)$ for T years at spot interest rate of r; for every $1 \leq k \leq n$ deposit I_k for t_k years at a spot rate of r_k.
- Wait T years: meanwhile make the payments due from the asset while withdrawing the corresponding deposits.
- At the end of this period the deposit balance is £$(S-I)e^{rT}$.
- Withdraw £$(S-I)e^{rT}$ from your deposit.
- Have the asset delivered for £F.
- Use it to close the short position on the asset.
- Pocket the difference $(S-I)e^{rT} - F > 0$.

□

3. Foreign exchange forward contracts

An interesting example of a forward contract is one in which the underlying asset is foreign currency– this is an example of a forward contract in which the underlying asset provides income continuously (in the form of continuously compounded interest on a deposit of foreign currency.) We will deal with these by introducing a new variant of no-arbitrage arguments which will be used repeatedly in this course: we will construct two portfolios whose values at a certain time in the future are known to be identical. We will then apply Theorem 3 to deduce that the spot values of these two portfolios are identical and this will allow us to find the correct no-arbitrage price.

THEOREM 6. *Consider now a forward contract on one unit of foreign currency whose spot rate is S i.e., it costs S units of domestic currency to purchase one unit of foreign currency. Let the maturity date be T years in the future, and let the yield curve in the foreign currency be $Y_f(t)$. Let F be the forward rate, i.e., the party with the short position in the forward contract will deliver in T years one unit of foreign currency and receive a payment of F units of domestic currency. Write $r = Y(T)$ and $r_f = Y_f(T)$. We have $F = Se^{(r-r_f)T}$.*

PROOF. Consider the following two portfolios:
Portfolio A: a long position in the forward contract and Fe^{-rT} units of domestic currency earning interest rate of r for T years.
Portfolio B: $e^{-r_f T}$ units of foreign currency earning interest rate r_f.

At time T portfolio A will generate F units of domestic currency which can be used to pay for the one unit of foreign currency. At time T portfolio B will also be worth one unit of foreign currency. Theorem 3 implies that both portfolios must have the same value at any time t with $0 \leq t \leq T$. If we consider $t = 0$, this gives us $Fe^{-rT} = Se^{-r_f T}$. □

4. Futures contracts

A *futures contract* is similar to a forward contract: it is also an agreement to deliver an asset at an agreed price F, the *futures price*, and at an agreed date, the *maturity date* or *delivery date*

Again, one can have a *short position* (a commitment to deliver the asset) or a *long position* (a commitment to buy the asset.)

Unlike forward contracts, which are private agreements and not traded, futures contracts are traded through brokers in specialised exchanges such as London's LIFFE and Chicago Board of Trade (CBOT). [2] These exchanges regulate both trading and delivery of assets. Underlying assets of futures contracts are standardized. [3]

One *closes out* a position in a futures contract by entering an identical contract but with the opposite position, e.g., to close out a long position in a March 2015 Cocoa futures, you take a short position on a March 2015 Cocoa futures.

Futures *are settled daily:* e.g., assume you took a short position in a May 2015 Brent crude futures, with futures price \$24.10, i.e., you agreed to provide 1,000 US barrels of petroleum of a certain quality for \$24.10 per barrel.

If at the end of the day the futures price for May 2015 Brent crude is \$24.20 then you lose \$0.10 per barrel, because now your barrel of oil can be delivered for \$0.10 more than what you will get. You pay your broker $\$1,000 \cdot 0.10 = \$100$ and your futures price is changed to \$24.20. If the next day sees a drop of \$0.20 of the May 2015 Brent crude your broker will credit you \$200 and your new futures price is changed to \$24. The daily settlement of futures is called *marking to market.*

The marking to market of futures contracts invalidates the no-arbitrage arguments used to produce forward prices. The reason for this is that the interest paid or obtained from the daily cash-flows generated by the futures contracts are *stochastic,* i.e., they are not known in advance.

However, one has the following.

Theorem 7 (The Futures-Forwards Equivalence Principle)). *If interest rates are deterministic then forward price and futures price coincide.*

For a proof, consult Chapter 3 of John Hull's *Options, Futures and other derivatives.*

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[2]Check, for example Oil futures prices from the NY mercantile exchange.

[3]Check, for example CME's wheat futures contract specification.

CHAPTER 3

Options

Definition. *Options* are contracts conferring certain rights regarding the buying or selling of assets. We call the asset referred by the option as the *underlying asset* of the option.

A *European call option* gives the owner the right to *buy* its underlying asset at a certain price *on* a certain date.

A *European put option* gives the owner the right to *sell* its underlying asset at a certain price *on* a certain date.

The date on which the owner of European options can exercise his right is called the *expiration date* and the price is called the *strike price.*

An *American call option* gives the owner the right to buy its underlying asset at a certain price *by* a certain date.

An *American put option* gives the owner the right to sell its underlying asset at a certain price *by* a certain date.

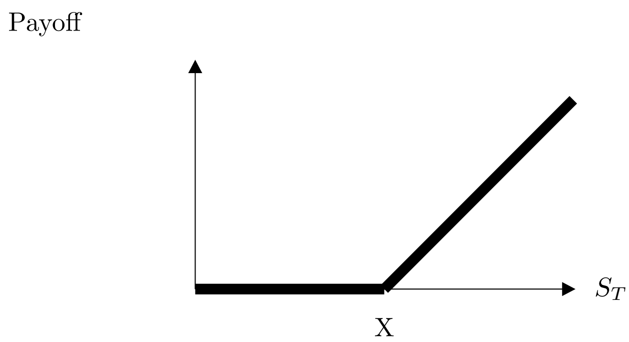
The date by which the owner of American options can exercise his right is called the *expiration date* and the price is called the *strike price.*

1. The payoffs of options

Consider a European call option with strike price X which has just expired at time $t = T$ and let S_T be the spot price of the underlying asset at the expiration of the option.

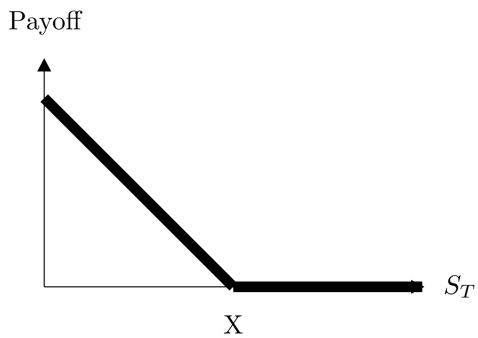
If $S_T > X$ the holder of the option will buy the asset for X and sell it for S_T, thus getting a payoff of $S_T - X$ from the option.

If $S_T \leq X$ the option holder will not exercise it; in this case the option generates no payoff.



Thus at expiration, the owner of the option will receive a payoff which is a function of the underlying asset price S_T at expiration, and that function is $\max\{S_T - X, 0\}$.

Similarly, the payoff of a corresponding European put option is given by $\max\{X - S_T, 0\}$ and is described by the following graph:



2. Elementary inequalities satisfied by option prices

We will now consider different option types on the same stock with spot price S, expiring in T years and with strike price X. We denote with c and p the current prices of European call and put options, and with C and P the current prices of American call and put options.

PROPOSITION 8.

(a) $c, C, p, P \geq 0$.
(b) $c \leq C \leq S$ *and* $p \leq P \leq X$.

PROOF. If any of the options has negative value $-v$, buy it for $-v$, i.e., receive the option plus an amount of v in cash, and forget about the option. If $c > C$, sell a European call option, buy an American call option, pocket $c - C > 0$ and wait for expiration. If the European option is exercised, exercise your American option and deliver the stock. A similar argument shows that $p \leq P$.

If $C > S$, sell the call option, buy the stock, pocket $C - S > 0$ and wait. If the option is exercised, deliver your stock, otherwise keep it. If $P > X$, sell the option and wait. If the option is exercised, buy the stock for X and pocket $P - X > 0$ plus the stock; if the option is not exercised pocket P. □ □

PROPOSITION 9. *Assume that the stock does not pay dividends and let r be the T-year spot interest rate.*

(a) $c > S - Xe^{-rT}$.
(b) $p > Xe^{-rT} - S$.

PROOF. To prove the first inequality consider:

Portfolio A: one European call option plus an amount of cash equal to Xe^{-rT} deposited for T years at an interest rate of r.

Portfolio B: one share.

After T years portfolio A will yield an amount of cash equal to X.

If, after T years, the stock price S_T is above X, the call option in portfolio A will be exercised, the share sold and the portfolio will be worth S_T. Otherwise, after T years, $S_T \leq X$, the option is not exercised and the portfolio will be worth X. So after T years portfolio A is worth $\max(S_T, X) \geq S_T$, and since portfolio B is always worth S_T after T years,

the initial value of portfolio A must be no less that the initial value of portfolio B, which is just S. But since sometimes portfolio A is worth more than portfolio B we have a strict inequality $c + Xe^{-rT} > S$.

To prove $p > Xe^{-rT} - S$ consider:

Portfolio C: one European put option plus one share.

Portfolio D: an amount of cash equal to Xe^{-rT} deposited for T years at an interest rate of r.

After T years portfolio D will be worth X.

If, after T years, $S_T < X$, then the put option in portfolio C will be exercised; the share is sold for X and the portfolio will be worth X. Otherwise, if , after T years, $S_T \geq X$, the option is not exercised and the portfolio will be worth S_T. So after T years portfolio C is worth $\max(S_T, X)$ and the initial value of portfolio C must be no less that the initial value of portfolio D which is just Xe^{-rT}. But since sometimes portfolio C is worth more than portfolio D we have a strict inequality $p + S > Xe^{-rT}$.

□

PROPOSITION 10 (Put-call parity). *Assume that the stock does not pay dividends and let r be the T-year spot interest rate. Then $c + Xe^{-rT} = p + S$.*

PROOF. Recall portfolios A and C above: After T years they are both worth $\max(S_T, X)$ where S_T is the stock price after T years. These two portfolios must have identical initial values, i.e., $c + Xe^{-rT} = p + S$. □

3. Optimal exercise for American style options

When should a rational investor exercise an American option?

PROPOSITION 11. *Assume that the stock does not pay dividends. The optimal exercise time for the American call option occurs at the expiration of the option and hence $c = C$.*

PROOF. For any time $0 < \tau < T$ write S_τ, c_τ, C_τ for the values at time τ years of the stock, European option and American option, respectively, and let r be the $T - \tau$ spot interest rate at time τ. We have

$$C_\tau \geq c_\tau > S_\tau - Xe^{-r(T-\tau)} > S_\tau - X,$$

where the second inequality follows from Proposition 9. but $S_\tau - X$ is the payoff, at time τ, from the exercise of the American option, and since the value of the American option exceeds that of this payoff, it should not be exercised. The optimal exercise of the American option will produce the same payoff as the European option, hence $c = C$. □

American *put* options may have an early optimal exercise date: e.g., suppose that on June 25th, 2002 you held American put options on Worldcom [1] stock with strike price of \$65 expiring in September 2002. Since you bought the stock the company disclosed that it inflated profits for over a year by improperly accounting for more than \$3.9 billion and the stock now sells for \$0.20. The payoff from exercising your option now would be \$64.8, almost its theoretical maximum. Things can only get worse as time progresses and you should exercise your options now.

PROPOSITION 12.

$$S - X < C - P < S - Xe^{-rT}.$$

PROOF. Since $P \geq p$ always, and since some of the time the payoff of the American option will be greater than that of the corresponding European option, $P > p$.

The second inequality is a consequence of Put-Call Parity, $P > p$ and $c = C$.

To prove $S - X < C - P$ consider:

[1] A particularly crooked American corporation which went bankrupt in the 00's.

Portfolio E: one European call option plus an amount of cash equal to X deposited for T years at an interest rate of r.

Portfolio F: one American put option plus one share.

At the time of expiration, portfolio E will be worth

$$\begin{aligned}\max(S_T - X, 0) + Xe^{rT} &= \max(S_T, X) - X + Xe^{rT} \\ &= \max(S_T, X) + X(e^{rT} - 1) \\ &> \max(S_T, X)\end{aligned}$$

and, if the American option has not been exercised before, portfolio F will be worth $\max(X - S_T, 0) + S_T = \max(S_T, X)$. So portfolio E expires with higher value than portfolio F.

If the American option was (rationally) exercised at an earlier time $0 \leq \tau < T$, then at that time portfolio F was worth $(X - S_\tau) + S_\tau = X$. At that time, the value of the cash in portfolio E is at least X. Since in either case there is always a time at which portfolio E is more valuable than portfolio F, the present value of portfolio E is greater than the present value of portfolio F, i.e., $C + X > P + S$.

□

4. Parameters affecting the prices of options

Suppose you hold a call option on a stock whose present price is £10 expiring in T years. What should happen to the price c of the call option if the stock price goes up to £15? The payoff of the option at expiration is $\max(S_T - X, 0)$ where S_T is the price of the stock at expiration. The rise in the stock price suggests that the market expects S_T to be higher as well, so *c and C are increasing functions of S.* For similar reasons, *p and P are decreasing functions of S.* Obviously, c and C are decreasing functions of X while p and P are increasing functions of X.

Suppose that the strike price of your call option is £20. If you are told that the variability of the stock price is very small, your option is not worth much because when the option expires the stock price is likely to be very close to £10, far below the strike price.

In general, share price movements can result in both higher and lower payoffs from an option, but the downside is limited (to losing the price paid for the option) while the upside is unlimited (in the case of a call option) or bounded by the strike price which is usually much higher than the option price. So we expect that *c, C are increasing functions of the volatility of the stock price.*

It is reasonable to assume that for longer expiration times T, the value of the stock at time T will have more variability. But on the other hand, for bigger T, the payoff of the option has to be discounted by a smaller discount factor. Obviously, *C and P are increasing functions of T.*

CHAPTER 4

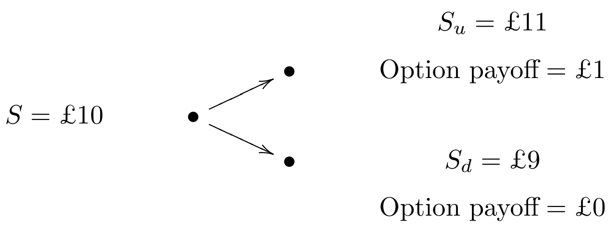
Binomial trees and risk neutral valuation

A *derivative* is an asset whose value depends on the value of another asset, e.g., Call/Put European/American options. In this chapter we find prices of derivatives providing a single payoff at a future date when the underlying asset price evolves in a particularly simple way.

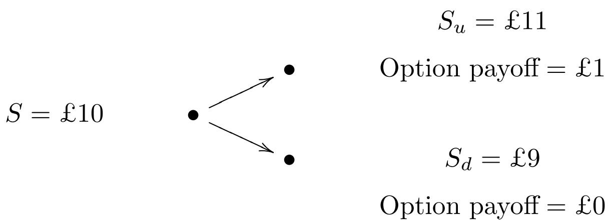
1. A motivating example

Consider a 1-year European call option on a stock with strike price £10. Assume that the current price of the stock is $S_0 = £10$ and that at the end of the one year period the price of the stock will be either $S_u = £11$ or $S_d = £9$. Assume further that the 1-year interest rate is 5%.

We picture this world as follows:



What should the price c of the option be? Consider a portfolio with δ shares of this stock, and short in one option.



If the stock price goes up the portfolio will be worth $11\delta - 1$ and if the stock price goes down it will be worth 9δ. What if we choose our δ so that

$$11\delta - 1 = 9\delta, \text{ i.e., } \delta = \frac{1}{2} \text{ ?}$$

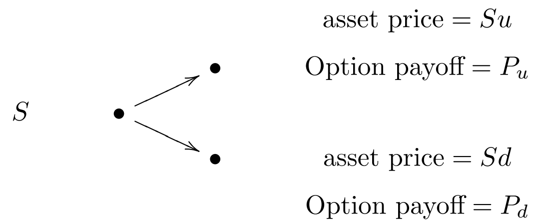
The value of this portfolio is the same in all possible states of the world! The portfolio must have a present value equal to its value in one year discounted to the present, i.e., $\frac{9}{2}e^{-0.05\times 1}$ but the current price of stock in the portfolio is £10/2, so

$$\frac{9}{2}e^{-0.05} = 5 - c \Rightarrow c = 5 - \frac{9}{2}e^{-0.05} \approx 0.72.$$

The probability of up or down movements in the stock price plays no role whatsoever!

2. Derivatives in a simple, up-down world

We generalise: consider a financial asset which provides no income and a financial derivative on that asset providing a single payoff t years in the future. The current price of the asset is S in t years the price of the stock will be either Su $(u > 1)$, resulting in a payoff of P_u from the derivative, or Sd $(0 \leq d < 1)$ resulting in a payoff of P_d from the derivative. Let r be the t-year interest rate.



Construct a portfolio consisting of δ units of the asset and -1 units of the derivative and choose δ so that the value of the portfolio after t years is certain: δ must satisfy

$$\delta Su - P_u = \delta Sd - P_d \Rightarrow \delta = \frac{P_u - P_d}{S(u-d)}.$$

The value of the portfolio in t years will be

$$\frac{P_u - P_d}{S(u-d)} Su - P_u = \frac{P_u - P_d}{u-d} u - P_u$$

and its present value is

$$e^{-rt}\left(\frac{P_u - P_d}{u-d} u - P_u\right).$$

Let x be the price of the derivative. We must have the following equality of present values

$$e^{-rt}\left(\frac{P_u - P_d}{u-d} u - P_u\right) = \delta S - x = \frac{P_u - P_d}{S(u-d)} S - x = \frac{P_u - P_d}{u-d} - x$$

Solving for x we obtain

$$x = \frac{P_u - P_d}{u-d} - e^{-rt}\left(\frac{P_u - P_d}{u-d} u - P_u\right) = \frac{e^{-rt}}{u-d}\left((e^{rt} - d)P_u + (u - e^{rt})P_d\right)$$

and if we let $q = \dfrac{e^{rt} - d}{u-d}$ we can rewrite x as $x = e^{-rt}\left(qP_u + (1-q)P_d\right)$. Notice: $0 \leq q = \frac{e^{rt}-d}{u-d} \leq 1$ and we can interpret q as a probability;. Now *In a world where the probability of the up movement in the asset price is q, the equation* $x = e^{-rt}\left(qP_u + (1-q)P_d\right)$ *says that the price of the derivative is the expected present value of its payoff.* Using these probabilities, stock price at time t has expected value

$$E = qSu + (1-q)Sd = qS(u-d) + Sd = \frac{e^{rt} - d}{u-d} S(u-d) + Sd = (e^{rt} - d)S + Sd = e^{rt}S,$$

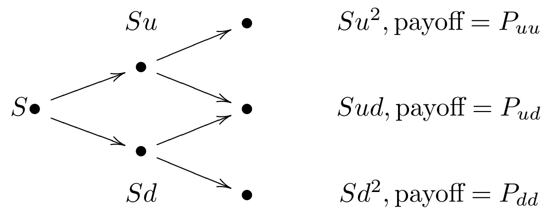
i.e., *the world where the probability of the up movement in the asset price is q is one in which the stock price grows on average at the risk-free interest rate.*

So in this world investors are indifferent to risk (unlike real-life investors).

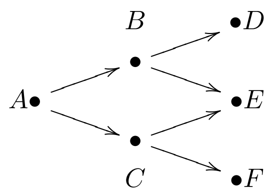
We refer to the probabilities q and $1-q$ as *risk neutral probabilities* and to equation above as a *risk neutral valuation.*

3. A two step process

Consider now a world where the price of the underlying changes twice, each time by either a factor of $u > 1$ or $d < 1$. After two periods the stock price will be Su^2, $Sud = Sdu$ or Sd^2. The derivative expires after the two periods producing payoffs of P_{uu}, $P_{ud} = P_{du}$ and P_{dd} respectively. Assume also each period is Δt years long and that interest rates for all periods is r.

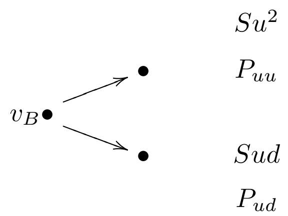


To find x, the value of the derivative, we now work our way backwards, from the end of the tree (i.e., the end of the second period) to the root (i.e., the present.)



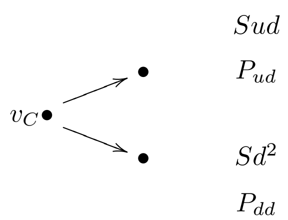
The value of the derivative is known at vertices D,E and F; these are the payoffs P_{uu}, P_{ud} and P_{dd}. How about nodes B and C?

We can find the value of the derivative at node B by considering the following one period tree:



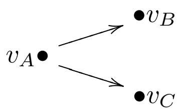
The risk neutral probability q of an upward movement is given by $q = \dfrac{e^{r\Delta t} - d}{u - d}$ and so the value v_B of the derivative at node B is $v_B = e^{-r\Delta t}\left(qP_{uu} + (1-q)P_{ud}\right)$.

Similarly, the value of the derivative at node C is obtained from



$$v_C = e^{-r\Delta t}\left(qP_{ud} + (1-q)P_{dd}\right).$$

Now we can work our way back one more step to node A;



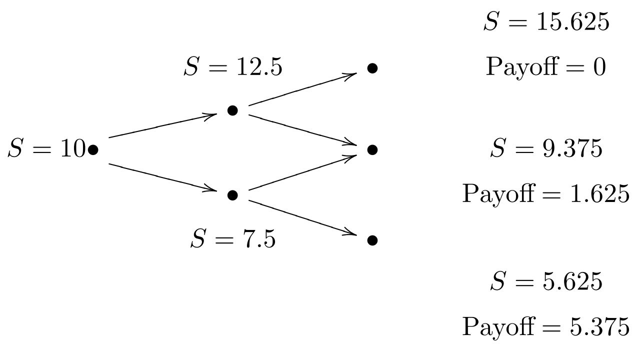
and the value at node A is

$$v_A = e^{-r\Delta t}\left(qv_B + (1-q)v_C\right).$$

4. An example

Consider a European put option on stock currently traded at £10, with strike price £11 and expiring in one year. Interest rates for all periods are 4%.

We use a two 6-month-period tree with $u = 5/4$ and $d = 3/4$ to estimate the price of the option.



Risk neutral probability of an upward movement of stock price

$$q = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.04/2} - 3/4}{5/4 - 3/4} \approx 0.5404.$$

Value v_B of the derivative at node B

$$v_B = e^{-r\Delta t}\left(qP_{uu} + (1-q)P_{ud}\right) \approx 0.7321,$$

value of the derivative at node C

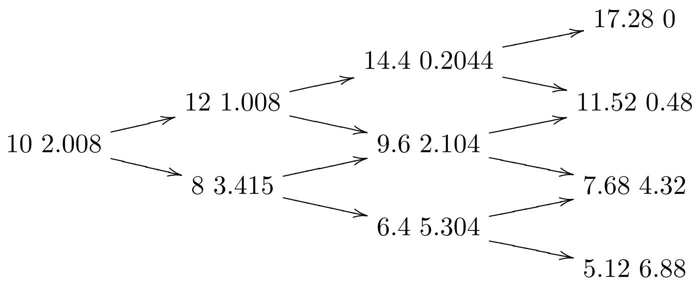
$$v_C = e^{-r\Delta t}\left(qP_{ud} + (1-q)P_{dd}\right) \approx 3.282,$$

value at node A

$$v_A = e^{-r\Delta t}\left(qv_B + (1-q)v_C\right) \approx 1.866$$

5. n-step trees

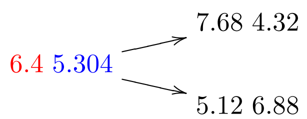
Consider a 18-month European put option with strike £12 on a stock whose current price is £10. Assume that interest rates for all periods are 5%. Use $u = 6/5$ and $d = 4/5$ to construct the following three step binomial tree.



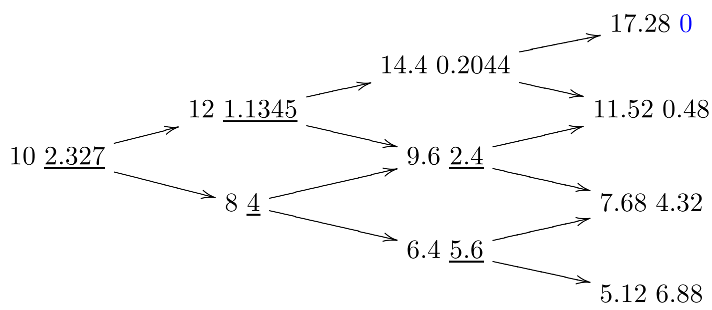
$$q = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05/2} - 4/5}{6/5 - 4/5} \approx 0.5633.$$
$$e^{-0.05/2}(0.5633 \times 0 + 0.4367 \times 0.48) \approx 0.2044$$
$$e^{-0.05/2}(0.5633 \times 0.48 + 0.4367 \times 4.32) \approx 2.104$$
$$e^{-0.05/2}(0.5633 \times 4.32 + 0.4367 \times 6.88) \approx 5.304$$
$$e^{-0.05/2}(0.5633 \times 0.2044 + 0.4367 \times 2.104) \approx 1.008$$
$$e^{-0.05/2}(0.5633 \times 2.104 + 0.4367 \times 5.304) \approx 3.415$$
$$e^{-0.05/2}(0.5633 \times 1.008 + 0.4367 \times 3.415) \approx 2.008$$

6. Example: an American option

Consider a 18-month American put option with strike £12 on a stock whose current price is £10. Assume that interest rates for all periods are 5%. Use $u = 6/5$ and $d = 4/5$ to construct a three step binomial tree. Consider the "dd" node in the previous figure.
Immediate exercise gives payoff of $12 - 6.4 = 5.6 > 5.304$ and that is the value of the option at this node.



The modified tree for the American option is then



(Underlined values differ from the European style case.)

$$q = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05/2} - 4/5}{6/5 - 4/5} \approx 0.5633.$$
$$e^{-0.05/2}(0.5633 \times 0 + 0.4367 \times 0.48) \approx 0.2044$$
$$e^{-0.05/2}(0.5633 \times 0.48 + 0.4367 \times 4.32) \approx 2.104, 12 - 9.6 = 2.4 > 2.104$$
$$e^{-0.05/2}(0.5633 \times 4.32 + 0.4367 \times 6.88) \approx 5.304, 12 - 6.4 = 5.6 > 5.304$$
$$e^{-0.05/2}(0.5633 \times 0.2044 + 0.4367 \times 2.4) \approx 1.1345$$
$$e^{-0.05/2}(0.5633 \times 2.4 + 0.4367 \times 5.6) \approx 3.7037, 12 - 8 = 4 > 3.7037$$
$$e^{-0.05/2}(0.5633 \times 1.1345 + 0.4367 \times 4) \approx 2.327$$

7. Using binomial trees for approximating values of derivatives

The assumption that the price of the asset underlying a derivative changes at a finite number of moments can approximate reality only if we allow the price to change at a large number of points in time, many more than two or three. This can lead to n-step trees for large values of n. These will contain $1 + 2 + 3 + \cdots + n = n(n+1)/2$ nodes and for even a modest value of n, say $n = 10$, these computations are best left to computers. In the case of certain exotic derivatives their value depends not only on the final price of an asset but on its history as well. These derivatives require even larger binomial trees.

Which values of u and d shall we use for these numerical approximations? Different choices would result in different prices for derivatives! The common practice is to take $d = 1/u$ and $u = e^{\sigma\sqrt{\Delta t}}$ where σ is the yearly standard deviation of the logarithm of the stock price and Δt is the length in years of every step in the tree. [1]

[1] One can show that this implies that the logarithm of stock prices Δt years in the future are normally distributed random variables with standard deviation $\sigma\sqrt{\Delta t}$. This will conform to our model of the evolution of share prices described in the next chapter

CHAPTER 5

The stochastic process followed by stock prices

Prices S_t of a an asset at a future time t is uncertain and we model it as a random variable. In this chapter we ask the question "what sort of random variable are S_t for $t \geq 0$?" Our answer to this question will consist of model involving Brownian motion as a major ingredient.

1. Brownian motion

DEFINITION. A *Brownian motion* is a family of random variables

$$\{B_t \mid t \geq 0\}$$

on some probability space $(\Omega, \mathcal{F}, P)$ such that:
(1) $B_0 = 0$,
(2) for $0 \leq s < t$ the increment $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$,
(3) for any $0 \leq t_1 < t_2 < \cdots < t_n$ the increments

$$B_{t_1} - B_0, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$$

are independent random variables, and
(4) For any $\omega \in \Omega$ the function $t \mapsto B_t(\omega)$ is continuous.

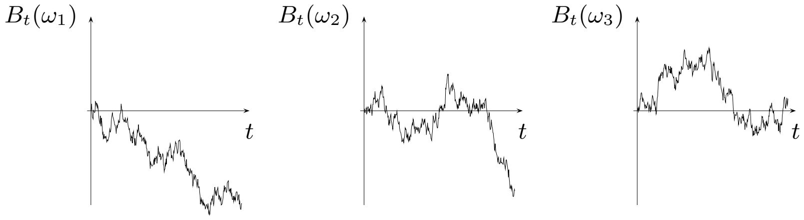


FIGURE 1. Three instances of Brownian Motion corresponding to $\omega_1, \omega_2, \omega_3 \in \Omega$.

Brownian motion exists (this statement is *Kolmogorov's Existence Theorem*) and has surprising properties. For example,

(a) The function $t \mapsto B_t(\omega)$ is nowhere differentiable with probability 1,
(b) if $B_t = x$ for some t then for any $\epsilon > 0$ the set $\{\tau : |\tau - t| < \epsilon \text{ and } B_\tau = x\}$ is infinite with probability one.

Brownian motion is useful for describing the jiggling of prices: buying and selling jiggle prices.

2. The Ito integral

Brownian motion is an example of a *stochastic process* i.e., a family of random variables indexed by time $t \geq 0$. We now construct a more general kind of stochastic processes whose definition is based on Brownian motion as follows.

We want to construct a stochastic process $\{X_t \mid t \geq 0\}$ on the same probability space $(\Omega, \mathcal{F}, \mathrm{P})$ on which Brownian motion $\{B_t \mid t \geq 0\}$ is defined with the property that the change of X over an infinitesimal period of time dt is given by

$$dX = a(\omega, t)dt + b(\omega, t)dB$$

where a and b are themselves stochastic processes on $(\Omega, \mathcal{F}, \mathrm{P})$ with continuous paths and where dB is the change in the Brownian motion over the infinitesimal period of time dt.

Fix any $\omega \in \Omega$; for any partition $\mathcal{P}$ of $[0,t]$ into small intervals $[s_0, s_1], [s_1, s_2], \ldots, [s_{n-1}, s_n]$ where $s_0 = 0$ and $s_n = t$, we compute the sum

$$\Sigma_{\mathcal{P}} = \sum_{i=0}^{n-1} b(\omega, s_i) \left(B(\omega)_{s_{i+1}} - B(\omega)_{s_i} \right).$$

Define the *norm* of the partition $\mathcal{P}$ to be

$$\|\mathcal{P}\| = \max \{s_1 - s_0, s_2 - s_1, \ldots, s_n - s_{n-1}\};$$

If b satisfies some technical conditions, the limit as $\|\mathcal{P}\| \to 0$ of $\Sigma_{\mathcal{P}}$ exists.

This limit is known as an *Ito integral* and we denote it with

$$\int_0^t b(\omega, s)\, dB(\omega)_s.$$

We now define the process

$$X_t(\omega) = X_0(\omega) + \int_0^t a(\omega, s)\, ds + \int_0^t b(\omega, s)\, dB(\omega)_s$$

for all $t \geq 0$.

Example. We calculate $\int_0^t s\, dB_s$ from first principles.

(1) Take any partition $\mathcal{P}$ of $[0,t]$ into small intervals $[s_0, s_1], [s_1, s_2], \ldots, [s_{n-1}, s_n]$ where $s_0 = 0$ and $s_n = t$.

(2) The sum

$$\Sigma_{\mathcal{P}} = \sum_{i=0}^{n-1} s_i \left(B_{s_{i+1}} - B_{s_i} \right)$$

is a sum of independent normally distributed random variables with mean 0 and variance $s_0^2(s_1 - s_0), s_1^2(s_2 - s_1), \ldots, s_{n-1}^2(s_n - s_{n-1})$.

(3) $\Sigma_{\mathcal{P}}$ is normally distributed with mean 0 and variance $s_0^2(s_1 - s_0) + s_1^2(s_2 - s_1) + \cdots + s_{n-1}^2(s_n - s_{n-1})$.

(4) As $\|\mathcal{P}\| \to 0$ this variance converges to $\int_0^t s^2 ds = t^3/3$.

(5) We conclude that $\int_0^t s dB_s$ is a normally distributed random variable with mean 0 and variance $t^3/3$.

Henceforth we write

$$dX = a(X, t)\, dt + b(X, t)\, dB$$

to denote that fact that X is a stochastic process defined by

$$X_t(\omega) = X_0(\omega) + \int_0^t a(X_s(\omega), s)\, ds + \int_0^t b(X_s(\omega), s)\, dB(\omega)_s.$$

We shall refer to stochastic processes of this form as *Ito processes.*

3. Modelling stock prices

As a first approximation we model the proportional increase in stock prices as a Brownian motion. We could then derive the following discrete time version

$$\frac{dS}{S} = \sigma dB$$

where dS is the change in the stock price over a short time from t to $t + dt$, $dB = B_{t+dt} - B_{dt}$ and B is a Brownian motion. (In particular proportional increases in S are independent, e.g., today's increase in a stock price is independent of tomorrow's increase.)

This model in unsatisfactory: it implies that the values of stocks vary without any long term trend, i.e., $E(\frac{dS}{S}) = 0$.

A quick glance at historical data shows that this is not very plausible: Despite the randomness

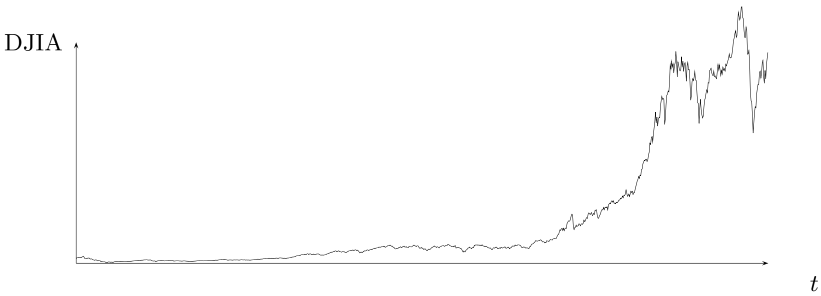


FIGURE 2. The Dow Jones Industrial Index

of the value of the DJIA, its long term exponential growth is quite visible.

To take into account this upward trend in stock prices we introduce a *drift term*

$$\frac{dS}{S} = \sigma dB + \mu \mathbf{dt}.$$

We refer to such a process S as a *geometric Brownian motion.*

Now

$$E(\frac{dS}{S}) = \mu dt.$$

We shall refer to μ as the *expected return* of the stock and to σ as the *volatility* of the stock.

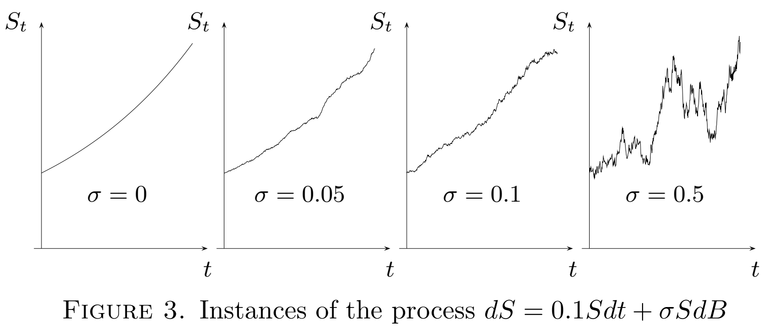


FIGURE 3. Instances of the process $dS = 0.1Sdt + \sigma SdB$

One can raise several objections to this model. Consider the following list of all trades in Vodafone shares between 16:22:08 and 16:23:08 on August 16th, 2002:

Trade time	Trade price	Bid	Ask	Volume	Block price	Buy/Sell
16:23:08	100.75p	100.75p	101p	180,614	£181,969	SELL
16:23:08	100.75p	100.75p	101p	1,185	£1,194	SELL
16:23:08	100.75p	100.5p	100.75p	250,000	£251,875	BUY
16:23:01	100.5p	100.25p	100.75p	2,500	£2,512	
16:22:55	100.75p	100.5p	100.75p	28,383	£28,596	BUY
16:22:55	100.75p	100.5p	100.75p	3,000	£3,022	BUY
16:22:55	100.75p	100.5p	100.75p	248,815	£250,681	BUY
16:22:55	100.75p	100.5p	100.75p	179,802	£181,151	BUY
16:22:55	100.75p	100.5p	100.75p	40,000	£40,300	BUY
16:22:42	100.5p	100.5p	100.75p	1,500	£1,508	SELL
16:22:41	100.75p	100.5p	100.75p	23,117	£23,290	BUY
16:22:39	100.527p	100.5p	100.75p	10,000	£10,053	SELL
16:22:38	100.75p	100.5p	100.75p	48,500	£48,864	BUY
16:22:38	100.75p	100.5p	100.75p	150,000	£151,125	BUY
16:22:38	100.75p	100.5p	100.75p	25,000	£25,188	BUY
16:22:38	100.75p	100.5p	100.75p	26,500	£26,699	BUY
16:22:29	100.688p	100.5p	100.75p	25,000	£25,172	BUY
16:22:18	100.75p	100.5p	100.75p	25,000	£25,188	BUY
16:22:08	100.75p	100.5p	100.75p	11,000	£11,082	BUY

(1) The model allows any real number to be a value for S, but in real life there is a smallest unit.
(2) We assume that prices are changing continuously but trades occur at discrete times.
(3) Shares are often traded through *market makers*. They buy at the *bid* price and sell at the *ask* price. So there are two prices! Sometimes it is crucial to model both.
(4) Sometimes, e.g., during market crashes, changes seem to have a "memory".

We should regard our model as an *approximation* to real life prices and trades, not as an accurate description.

4. Ito's Lemma.

Consider a stochastic process X_t whose change over a small interval of time from t to $t + dt$ is given by

$$dX = a(X,t)dt + b(X,t)dB$$

where $a(x,t)$ and $b(x,t)$ are functions of x and t. Consider a new stochastic process $Y = G(X,t)$ where $G(x,t)$ is a function of x and t. What sort of process is Y?

If B where just a variable rather than Brownian motion, the chain rule would give us

$$\begin{aligned} dY &= \left(\frac{\partial G}{\partial x}\frac{\partial X}{\partial t} + \frac{\partial G}{\partial t}\right) dt + \frac{\partial G}{\partial x}\frac{\partial X}{\partial B}\, dB \\ &= \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t}\right) dt + \frac{\partial G}{\partial x} b\, dB. \end{aligned}$$

However, *this is false*: B is not a variable, and it turns out that we have to add a term:

THEOREM 13 (Ito's Lemma). *Assume that $G(x,t)$ is twice continuously differentiable with respect to x and continuously differentiable with respect to t. The process $Y = G(X,t)$ is also an Ito* *process. In fact,*

$$dY = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right) dt + \frac{\partial G}{\partial x}b \; dB.$$

The reason why this theorem holds is that dB^2 is not negligible compared to dt and dB, in fact, roughly speaking "$dB^2 = dt$". To see this write $dB = B_{t+\Delta t} - B_t$ as $\sqrt{\Delta t}Z$ for some standard normal Z. Note that $1 = \mathrm{Var}(Z) = \mathrm{E}(Z^2) - \mathrm{E}(Z)^2 = \mathrm{E}(Z^2)$, hence $\mathrm{E}(dB^2) = \mathrm{E}(\Delta t Z^2) = \Delta t\,\mathrm{E}(Z^2) = \Delta t$, and $\mathrm{Var}(dB^2) = \Delta t^2\,\mathrm{Var}(Z^2)$ is of order Δt^2.

As $\Delta t \to 0$, $\mathrm{Var}(dB^2) \ll \Delta t = \mathrm{E}(dB^2)$, so dB^2 "converges to" Δt.

Here is a sketch of a (very) informal proof of Ito's Lemma:

Using the Taylor series for G we can write

$$dY = \frac{\partial G}{\partial x}\Delta X + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta X^2 + \frac{\partial^2 G}{\partial x \partial t}\Delta X \Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \text{ higher order terms.}$$

Now

$$(\Delta X)^2 = (a\,\Delta t + b dB)^2 = b^2 \Delta t + \text{ higher terms in } \Delta t$$

So for small Δt we approximate

$$dY \approx \frac{\partial G}{\partial x}\Delta X + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\Delta t + \text{ higher terms in } \Delta X, \Delta t =$$
$$\frac{\partial G}{\partial x}(a\Delta t + b\Delta B) + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\Delta t + \text{ higher terms in } \Delta B, \Delta t.$$

5. The stochastic process followed by forward stock prices

Consider a forward contract on stock paying no dividends maturing at time T; let $F(t)$ be its forward price at time $t \geq 0$:

$$F(t) = S(t)e^{r(T-t)},$$

where $S(t)$ is the spot price of the stock at time t. Regard F as a function of s and t, i.e., $F = F(s,t) = se^{r(T-t)}$:

$$\frac{\partial F}{\partial s} = e^{r(T-t)}, \quad \frac{\partial^2 F}{\partial s^2} = 0 \quad \text{and} \quad \frac{\partial F}{\partial t} = -rse^{r(T-t)}.$$

Our model assumes that

$$dS = \mu S dt + \sigma S dB$$

so Ito's Lemma implies that

$$dF = \left(e^{r(T-t)}\mu S - rSe^{r(T-t)}\right) dt + e^{r(T-t)}\sigma S dB = (\mu - r)F dt + \sigma F dB,$$

i.e., F follows a geometric Brownian motion with drift $\mu - r$.

6. The stochastic process followed by the logarithm of stock prices

Let S be the spot price of a certain stock at time t and let $G = G(s,t) = \log s$.

Since

$$\frac{\partial G}{\partial s} = \frac{1}{s}, \quad \frac{\partial^2 G}{\partial s^2} = \frac{-1}{s^2} \quad \text{and} \quad \frac{\partial G}{\partial t} = 0$$

and since $dS = \mu S dt + \sigma S dB$

Ito's Lemma implies that

$$dG = \left(\frac{1}{S}\mu S - \frac{\sigma^2 S^2}{2S^2}\right) dt + \frac{\sigma S}{S} dB = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dB.$$

Corollary 14. *Consider a fixed time T in the future and the spot price S_T at that time: the logarithm of the proportional price change of the stock,*

$$\log \frac{S_T}{S} = \log S_T - \log S,$$

is normally distributed with expected value $\left(\mu - \frac{\sigma^2}{2}\right) T$ *and variance* $\sigma^2 T$.

CHAPTER 6

The Black-Scholes pricing formulas

In this chapter we derive formulas for the prices of European call and put options.

1. The Black-Scholes differential equation

LEMMA 15. *Assume that a stock price S follows the Geometric Brownian motion*

$$dS = \mu S\,dt + \sigma S\,dB$$

where μ and σ are constants.
Let $f = f(S,t)$ be the value at time t of any derivative contingent on the value of S at some $t = T$. Assume $f(s,t)$ is twice differentiable with respect to s and differentiable with respect to t. The process followed by f is

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial f}{\partial S}\sigma S dB.$$

PROOF. Apply Ito's Lemma with $a(S,t) = \mu S$ and $b(S,t) = \sigma S$. □

The discrete version of the equation is

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) \Delta t + \frac{\partial f}{\partial S}\sigma S \Delta B.$$

where ΔB is a normally distributed random variable with zero mean and variance Δt
Consider a portfolio consisting of a variable quantity $\frac{\partial f}{\partial S}$ of shares and -1 derivatives; let Π be the value of this portfolio, i.e., $\Pi = \frac{\partial f}{\partial S}S - f$.

After a short period of time Δt the value of the portfolio changes by

$$\Delta\Pi = \frac{\partial f}{\partial S}\Delta S - \Delta f = \frac{\partial f}{\partial S}\left(\mu S \Delta t + \sigma S \Delta B\right) - \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t - \frac{\partial f}{\partial S}\sigma S \Delta B = \left(-\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 - \frac{\partial f}{\partial t}\right)\Delta t$$

Notice that $\Delta\Pi$ is non-stochastic!

Since the value of Π in the future is known with certainty, its value must be increasing at the same rate as a risk-free deposit earning interest r:

$$\Delta\Pi = e^{r\Delta t}\Pi - \Pi$$

and for infinitesimal Δt we obtain

$$\Delta\Pi \approx (1 + r\Delta t)\Pi - \Pi \Rightarrow \Delta\Pi \approx r\Pi\Delta t.$$

If we substitute equations back we obtain the *Black-Scholes differential equation:*

THEOREM 16.

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

2. Boundary conditions

To obtain prices from the Black-Scholes PDE we impose *boundary conditions*, e.g., for a European call option with strike X and expiring at time T the boundary condition is $f(S,T) = \max\{S - X, 0\}$ for all S. For a European put option with strike X and expiring at time T the boundary condition is $f(S,T) = \max\{X - S, 0\}$ for all S.

The theory of linear partial differential equations shows that we obtain a *unique* solution by imposing a further boundary condition at $S = 0$. Now if $S_t = 0$ at any time $0 \le t \le T$, $S_t = 0$ for all $t \ge T$ and hence $f(0,t) = e^{-r(T-t)} f(0,T)$ which provides us with a boundary condition. E.g., in the case of European call options, $f(0,t) = 0$. (For numerical purposes we may impose boundary conditions at $S = \infty$).

Example. $f(S,t) = e^{rt} S^{1-2r/\sigma^2}$ is a solution of the Black-Scholes PDE

$$\frac{\partial f}{\partial t} = rf(S,t), \quad \frac{\partial f}{\partial S} = (1 - \frac{2r}{\sigma^2}) e^{rt} S^{-2r/\sigma^2}, \quad \frac{\partial^2 f}{\partial S^2} = -(1 - \frac{2r}{\sigma^2}) \frac{2r}{\sigma^2} e^{rt} S^{-1-2r/\sigma^2}$$

and

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} =$$
$$rf(S,t) + rS(1 - \frac{2r}{\sigma^2}) e^{rt} S^{-2r/\sigma^2} - \frac{1}{2}\sigma^2 S^2 (1 - \frac{2r}{\sigma^2}) \frac{2r}{\sigma^2} e^{rt} S^{-1-2r/\sigma^2} =$$
$$rf(S,t) + r(1 - \frac{2r}{\sigma^2}) e^{rt} \left[S \times S^{-2r/\sigma^2} - S^2 \times S^{-1-2r/\sigma^2} \right] = rf(S,t) + 0.$$

Consider a derivative on certain stock with single payoff at time $T > 0$ amounting to S_T^{1-2r/σ^2} where r is the constant interest rate. Assume that $1 - 2r/\sigma^2 > 0$. Find the value of the derivative at time $0 \le t \le T$.

Consider a portfolio consisting of e^{rT} derivatives and write $v(S,t)$ for the price of this portfolio; we have $v(S_T, T) = e^{rT} S_T^{1-2r/\sigma^2} = f(S_T, T)$. Now,

- Both $v(S,t)$ and $f(S,t)$ are solutions of the Black-Scholes PDE, and
- v and f have the same values at the boundary: $v(0,t) = f(0,t) = 0$, and $v(S,T) = f(S,T)$.

But there is only one solution of the Black-Scholes PDE satisfying a given boundary condition and this forces $v(S,t) = f(S,t) = e^{rt} S^{1-2r/\sigma^2}$.

So the price of one derivative is $e^{-rT} e^{rt} S^{1-2r/\sigma^2} = e^{-r(T-t)} S^{1-2r/\sigma^2}$.

3. The Black-Scholes pricing formulas

Theorem 17 (The Black-Scholes pricing formulas). *Consider a European option at time t on stock with spot price S, with strike price X and expiring at time T.*

Let σ be the annual volatility of the stock, and r the T-year interest rate. Define

$$d_1 = \frac{\log(S/X) + \left(r + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\log(S/X) + \left(r - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

Then the price of the call option at time t is

$$c = S\Phi(d_1) - Xe^{-r(T-t)}\Phi(d_2)$$

and the price of the put option is

$$p = Xe^{-r(T-t)}\Phi(-d_2) - S\Phi(-d_1).$$

where Φ is the standard normal distribution function.

Proof. Define the function $c(S,t) = S_t\Phi(d_1) - Xe^{-r(T-t)}\Phi(d_2)$. In view of the BS differential equation we need to verify that

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} = rc,$$
$$\lim_{t\to T} c(S_t,t) = \max\{S_T - X, 0\}$$
$$\lim_{S\to 0} c(S,t) = 0.$$

The verification of both these statements is straightforward (but tedious!) A similar argument produces the value of p. □

The Black-Scholes pricing formulas are a result of a no-arbitrage argument: if violated use portfolio Π to get a free lunch. In practice one cannot adjust Π continuously, and if there are trading charges, one cannot even make frequent adjustments.

4. The Black-Scholes pricing formulas: the risk neutral valuation approach.

Recall the BS PDE :

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf.$$

Notice that μ does not appear here! The quantity $\mu - r$ is the excess return that investors demand when investing in an asset whose volatility measure is σ. We conclude that the risk-aversion of investors does not affect the value of derivatives given as solutions of the Black-Scholes PDE. Since this formula holds regardless of amount of risk-aversion and we might as well assume that investors are risk neutral.

Risk-neutral investors only care about the expected return of their investments, and they do not care about uncertainty regarding these returns. Hence all investments in a risk neutral world must have the same expected return r equal to the risk-free interest rate.

Definition (The principle of risk-neutral valuation). Let f be the price of a derivative which pays $H(S_T)$ for some function H at a future time T. In our risk-neutral world the stock price has expected return r, the risk-free T-year interest rate. In a risk-neutral world the current value of the derivative $f(S,0)$ is the present value of the expected value of the derivative payoff at time T, i.e.,

$$f(S,0) = e^{-rT}\widetilde{\mathrm{E}}\left(H\left(S_T\right)\right)$$

where $\widetilde{\mathrm{E}}$ denotes expected values in our risk-neutral world.

5. Digital options

Consider a derivative on a stock which at expiration time T pays £1 if $S_T \le a$, for some positive number a, and zero otherwise. (These options are known as *digital* or *binary* options.) Let the volatility of the stock price be σ and assume all interest rates are constant and equal to r. Apply a risk neutral valuation argument to show that, for any $0 \le t \le T$, the value of this derivative equals

$$e^{-r(T-t)}\Phi\left(\frac{\log(a/S) - \left(r - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

where Φ is the cumulative distribution function of the standard normal distribution.

We are assuming S follows the process

$$dS = \mu S dt + \sigma S dB$$

for constants μ and σ, and so at time $0 \le t \le T$, $\log S_T$ is normally distributed with mean $\log S + (\mu - \frac{\sigma^2}{2})(T-t)$ and standard deviation $\sigma\sqrt{T-t}$. In a risk neutral world we set $\mu = r$ and now $\log S_T$ is normally distributed with mean $\log S + (r - \frac{\sigma^2}{2})(T-t)$ and standard deviation $\sigma\sqrt{T-t}$.

The event $S_T \le a$ is equivalent to the event $\log S_T \le \log a$ and so the probability in this risk neutral world of the event $S_T \le a$ is $\Phi\left(\frac{\log a - (\log S + (r - \frac{\sigma^2}{2})(T-t))}{\sigma\sqrt{T-t}}\right) = \Phi\left(\frac{\log a/S - (r-\frac{\sigma^2}{2})(T-t))}{\sigma\sqrt{(T-t)}}\right)$. In our risk neutral world the value of the derivative is the present value of the expected value of its payoff, i.e.,

$$e^{-r(T-t)}\Phi\left(\frac{\log(a/S) - \left(r - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

6. Back to European call/put options

Our assumptions imply that $\log S_T$ is normally distributed with mean $\log S + \left(\mu - \dfrac{\sigma^2}{2}\right)T$ and variance $\sigma^2 T$; in our risk neutral valuation argument we set $\mu = r$. Risk-neutral investors also expect the current value of the derivative $f(S,0)$ to be the expected value of the present value of the derivative payoff at time T, i.e.,

$$f(S,0) = e^{-rT}\widetilde{E}\left(H\left(S_T\right)\right).$$

Consider the case where $H(y) = \max\{y - X, 0\}$ with X being the strike price of the call option.

We have $c = e^{-rT}\widetilde{E}\left(\max\{S_T - X, 0\}\right)$.

Let ϕ be the density function of the lognormal random variable S_T in our risk neutral world.

$$c = e^{-rT}\int_X^\infty (y - X)\phi(y)\,dy.$$

Lemma 18.

$$\int_X^\infty (y-X)\phi(y)\,dy = Se^{rT}\Phi\left(d_1\right) - X\Phi\left(d_2\right)$$

where $d_1 = \frac{\log(S/X) + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}}$, $d_2 = \frac{\log(S/X) + \left(r - \sigma^2/2\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$.

Now we obtain

$$c = S\Phi\left(d_1\right) - e^{-rT}X\Phi\left(d_2\right)$$

and a similar argument shows that the price of a European put option with strike price X is

$$p = Xe^{-rT}\Phi(-d_2) - S\Phi(-d_1)$$

7. Volatility

The parameters European options are the spot price of the stock, the strike price, time to expiration, the interest rate, and the volatility σ of the stock price.

The first four parameters are always known; but volatility is not directly observable. One can estimate *historical volatility* by analysing the time series consisting of prices of the stock at previous times in the past.

However, there are many problems involved with these estimates. One such problem is that even though our model for stock prices assumes constant volatility, in practice some periods of time, e.g., immediately after September 11th, 2001, are more volatile than others. So we might want to give lower weights to more distant measurements.

Traders do not normally use historical volatility when applying the Black-Scholes pricing formulas. Instead they use *implied volatilities* which are the value of the volatility parameter which will produce the observed market price of a given option. This sounds like a circular argument, but it is useful for example to produce prices of an option based on a similar one and to produce new prices as the price of the underlying stock changes or as time progresses.

CHAPTER 7

Portfolio Theory

In this chapter we aim to find optimal investment strategies. To do so we first need to understand investors' preferences, i.e., how does a person decide which investment is best?

Consider the following example involving three £1,000 one-year investments:

Portfolio A: Will be worth £1,100 with probability 1.

Portfolio B: Will be worth £1,000 with probability 1/2 and £1,300 with probability 1/2.

Portfolio C: Will be worth £500 with probability 1/10, £1,200 with probability 8/10 and £3,000 with probability 1/10.

The expected returns are

$$r_A = \frac{1100-1000}{1000} = 0.1,$$

$$r_B = \frac{1}{2}\frac{1000-1000}{1000} + \frac{1}{2}\frac{1300-1000}{1000} = 0.15$$

$$r_C = \frac{1}{10}\frac{500-1000}{1000} + \frac{8}{10}\frac{1200-1000}{1000} + \frac{1}{10}\frac{3000-1000}{1000} = 0.31$$

Now consider the following investors:

Mr. X: Wants to sail around the world on a cruise costing £1,200 a year from now.

Ms. Y: Must repay her mortgage in a year and must have £1,100 to do so.

Dr. Z: Needs to buy a rare book worth £1,300.

The chances of Mr X. sailing around the world if he invests in investments A,B or C are 0, 1/2 and 9/10 respectively, so he should be advised to invest in C. Only investment A guarantees a return sufficient for Ms. Y to pay her mortgage and she should choose it. The probability of the investments being worth at least £1,300 after a year are 0, 1/2 and 1/10 respectively, so Dr. Z should invest in portfolio B.

So different investors prefer different investments!

"highest expected returns"$\neq$"optimal": the *whole distribution* of the returns needs to be taken into account.

Instead of considering the whole distribution of the returns of an investment we will take into account two parameters:

- the *expected return* which we denote r, and
- the *standard deviation of the return* which we denote σ.

We now rephrase our problem: given a set of investments whose returns have known expected values and standard deviations, which one is "optimal"?

1. Axioms satisfied by preferences

Consider two investments A and B with expected returns r_A and r_B and standard deviation of returns σ_A and σ_B.

The following assumptions look plausible:

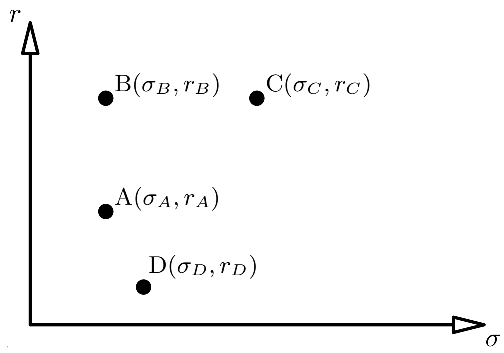


FIGURE 1. B is preferable to both A and C. Investments A and C are incomparable.

A1: Investors are greedy: If $\sigma_A = \sigma_B$ and $r_A > r_B$ investors prefer A to B.
A2: Investors are risk averse: If $r_A = r_B$ and $\sigma_A > \sigma_B$ investors prefer B to A.
A3: Transitivity of preferences: If investment B is preferable to A and if investment C is preferable to B then investment C is preferable to A.

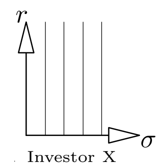
We are introducing a partial ordering $\prec$ on the points of the σ-r plane:

2. Indifference curves

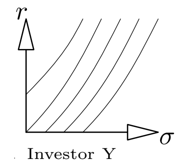
We describe the preferences of an investor by specifying the sets of investments which are equally attractive to the given investor. We do so by defining an *indifference curve* of an investor: this is a curve consisting of points (σ, r) for which investments with these expected returns and standard deviation of returns are all equally attractive to our investor.

Notice that assumptions A1, A2 and A3 imply that these curves must be non-decreasing.

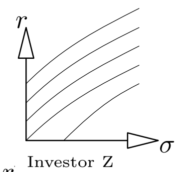
Consider hypothetical investors X,Y,Z and W with the following indifference curves.



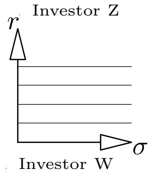
Investor X cannot tolerate uncertainty at all: this investor prefers a certain zero return rather a very large expected return with a small degree of uncertainty.



Investor Y is willing to take some risks



but not as much as investor Z
(his indifference curves are steeper.)



Investor W is risk neutral.

3. Portfolios consisting entirely of risky investments

Consider two investments A and B with expected returns r_A and r_B and standard deviation of returns σ_A and σ_B. We split an investment of £1 between the two investments: consider portfolio Π_t consisting of t units of investment A and $1-t$ units of investment B. We can do this for *any* t and not just $0 \le t \le 1$. For example, to construct portfolio Π_2 we short sell £1 worth of B and buy £2 worth of A, for a total investment of £1. Let A and B be the random variables representing the annual return of investments A and B.

The variance of Π_t is given by

$$\begin{aligned}
\operatorname{Var}(\Pi_t) &= \operatorname{Var}(tA + (1-t)B) \\
&= \operatorname{Var}(tA) + \operatorname{Var}((1-t)B) + 2\operatorname{Covar}(tA, (1-t)B) \\
&= t^2 \operatorname{Var}(A) + (1-t)^2 \operatorname{Var}(B) + 2t(1-t)\operatorname{Covar}(A, B) \\
&= t^2 \operatorname{Var}(A) + (1-t)^2 \operatorname{Var}(B) + 2t(1-t)\rho(A, B)\sqrt{\operatorname{Var}(A)\operatorname{Var}(B)} \\
&= (t\sigma_A)^2 + 2t(1-t)\rho(A, B)\sigma_A\sigma_B + ((1-t)\sigma_B)^2
\end{aligned}$$

The shapes of these curves are *concave*:

PROPOSITION 19. *The curve in the σ-r plane given parametrically by*

$$\left(\sqrt{(t\sigma_A)^2 + 2t(1-t)\rho(A, B)\sigma_A\sigma_B + ((1-t)\sigma_B)^2}, tr_A + (1-t)r_B\right)$$

for $0 \le t \le 1$ lies to the left of the line segment connecting the points (σ_A, r_A) and (σ_B, r_B).

PROOF. Since $\rho(A, B) \le 1$,

$$\begin{aligned}
\sqrt{(t\sigma_A)^2 + 2t(1-t)\rho(A, B)\sigma_A\sigma_B + ((1-t)\sigma_B)^2} &\le \sqrt{(t\sigma_A)^2 + 2t(1-t)\sigma_A\sigma_B + ((1-t)\sigma_B)^2} \\
&= \sqrt{(t\sigma_A + (1-t)\sigma_B)^2} \\
&= t\sigma_A + (1-t)\sigma_B.
\end{aligned}$$

The result follows from the fact that the parametric equation of the line segment connecting the points (σ_A, r_A) and (σ_B, r_B) is

$$\{(t\sigma_A + (1-t)\sigma_B, tr_A + (1-t)r_B) \mid 0 \le t \le 1\}.$$

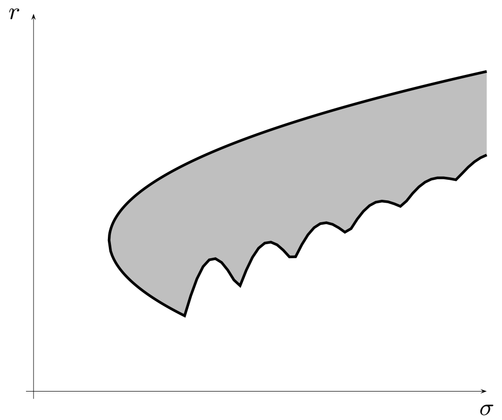
□

4. The feasible set

Suppose now that there are many different investments $A_1, \ldots, A_n$ available. We can invest our one unit of currency by investing t_i in A_i for each $1 \le i \le n$ as long as $\sum_{i=1}^n t_i = 1$. What are all possible pairs (σ, r) corresponding to these portfolios? This set of points in the σ-r plane is called the *feasible set.*

5. Efficient portfolios

We now return to the main question in this chapter: which portfolios among all possible ones should an investor satisfying axioms A1,A2 and A3 choose?



DEFINITION. An *efficient portfolio* is a feasible portfolio that provides the greatest expected return for a given level of risk, or equivalently, the lowest risk for a given expected return. (This is also called an optimal portfolio.)

The *efficient frontier* is the set of all efficient portfolios.

Obviously, our investor should choose a portfolio along the efficient frontier!

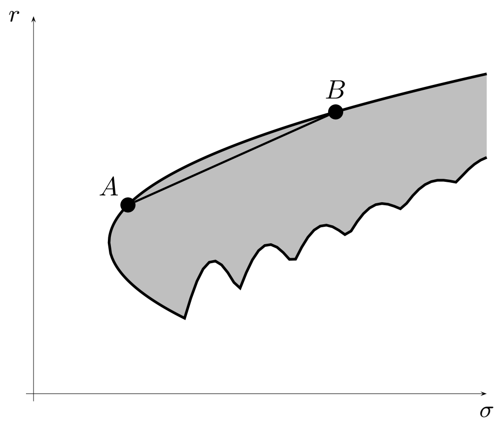


FIGURE 2. Feasible sets are convex along efficient frontier.

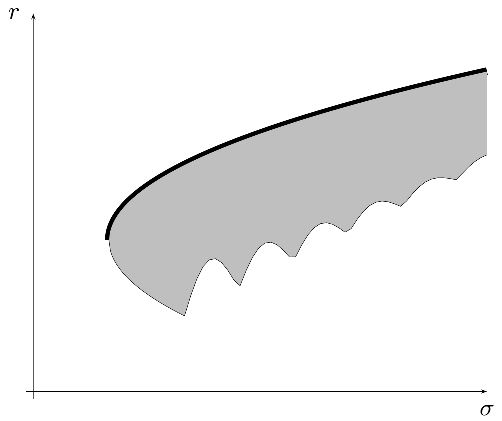
The feasible set is *convex* along the efficient frontier, in the sense that for any two portfolios A and B in the feasible set, there exist feasible portfolios above the portfolios in the segment connecting A and B.

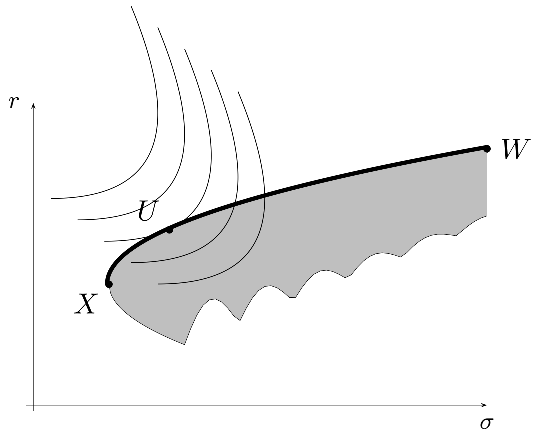
Which portfolio along the efficient frontier will our investor choose? This is where risk preferences start playing a role.

6. Different choices of portfolios for different appetites for risk

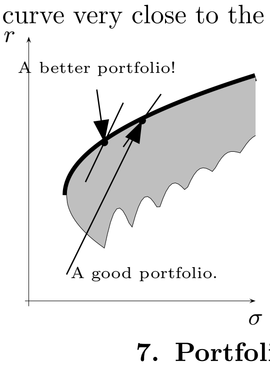
Consider investors X (with no risk tolerance at all) and W (risk neutral) discussed before together with investor U whose indifference curves are given below.

If the indifference curves are not too badly behaved, e.g., if the indifference curves are the level curves of some smooth function $F(\sigma, r)$, then we should expect the optimal portfolio to be at a point





where the indifference curve is tangent to the efficient frontier. Otherwise, if it occurs at a point where the indifference curve intersects the efficient frontier transversally, find an almost parallel indifference curve very close to the original one and to its left.



Portfolios on this curve are more desirable and, if we chose the second indifference curve close enough to the original one, it will also intersect the efficient frontier, and this intersection will correspond to a better choice of portfolio than the one corresponding to the original point of intersection.

7. Portfolios containing risk-free investments

We now add a risk-free investment B. Let r_B be its (expected) return. Since r_B is constant, its covariance with the returns of any other portfolio Π is zero so the portfolio Π_t consisting of t units of currency invested in B and $(1-t)$ units of currency invested in Π has expected return

$$E\left(tB + (1-t)\Pi\right) = tr_B + (1-t)r_\Pi$$

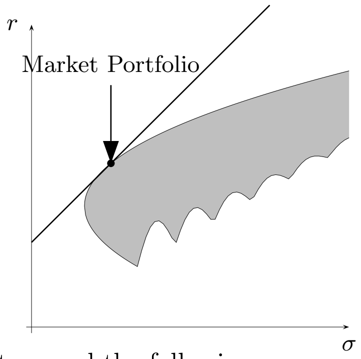
(where we used B and Π to denote also the returns of the investments B and Π) and standard deviation of return

$$\sqrt{Var(tB + (1-t)\Pi)} = \sqrt{Var((1-t)\Pi)} = \sqrt{(1-t)^2 Var(\Pi)} = |1-t|\sigma_\Pi.$$

The curve $t \mapsto (\sigma_{\Pi_t}, r_{\Pi_t})$ for $t \leq 1$ is a straight line passing through the points $(0, r_B)$ and (σ_Π, r_Π) and all the points on or below such a line will be part of the feasible set.

What happens to the efficient frontier? Consider the set S consisting of all the slopes s of lines ℓ_s in the σ-r plane which pass through the point $(0, r_B)$ and intersect the feasible set. Let m be the supremum of S. Consider now the line ℓ_m which is above all the others: The line ℓ_m will either be tangent to the efficient frontier or asymptotic to it.

(We will see in Chapter 8 that, if we impose additional conditions on markets and investors, ℓ_m cannot be an asymptote of the efficient frontier and so it is tangent to it.)



This point of tangency is called the *market portfolio* and we shall denote the corresponding portfolio with M.
The new efficient frontier, ℓ_m is called the *capital market line*.

We just proved the following:

THEOREM 20. *In the presence of a risk-free investment there exists an (essentially) unique investment choice consisting entirely of risky investments which is efficient, namely, the market portfolio.*

Any other efficient investment is a combination of an investment in the market portfolio and in the risk-free investment.

CHAPTER 8

The Capital Asset Pricing Model

In this chapter we aim to find the "correct" price of financial assets. In doing so we will introduce a new notion of correctness for prices: when we valued derivatives, correct prices were those which created no arbitrage opportunities; in this chapter prices are correct if they are stable.

We make additional assumptions about markets and investors:

- A4 Markets are in equilibrium: The total demand for any financial instrument equals its total supply.
- A5 Uniform horizon: All investors are investing for the same period of time.
- A6 Homogeneity: All investors agree on the expected returns of investments, their standard deviations of returns and the correlations between these returns. All investors can borrow and lend unlimited amounts of money at the same uniform risk-free rate.
- A7 No friction: There are no transaction costs and no taxes.

1. The market portfolio.

The market portfolio is the only efficient portfolio consisting entirely of risky investments. What is this portfolio?

THEOREM 21. *Let $I_1, I_2, \ldots, I_n$ be* all *risky investments, and assume their total market value is $w_1, w_2, \ldots, w_n$. Let $W = w_1 + w_2 + \cdots + w_n$. The market portfolio consists of a portfolio of investments of $\frac{w_j}{W}$ in I_j for each $1 \leq j \leq n$.*

PROOF. Assume that there are m investors and that for any $1 \leq k \leq m$ investor k holds h_{kj} worth of investment j.

Write $H_k = \sum_{j=1}^{n} h_{kj}$ for the total amount invested in risky investments for investor k. Axioms A1, A2, A3, the results of Chapter 7 and Axioms A5, A6 imply that all investors will have the same proportion of each risky investment, i.e., $\frac{h_{kj}}{H_k} = \frac{h_{lj}}{H_l}$ for every $1 \leq k, l \leq m$.

Axiom A4 implies that for each $1 \leq j \leq n$, $w_j = \sum_{k=1}^{m} h_{kj}$. (Notice w_j is the total supply of investment j while $\sum_{k=1}^{m} h_{kj}$ is its total demand; the two should be equal if the market is in equilibrium.) Summing over all investments gives $W = \sum_{k=1}^{m} H_k$.

Now the proportion of the market portfolio invested in I_j is the same as any investor's proportion of investment in I_j out of the total risky investments, so we can write this proportion as

$$\begin{aligned}\frac{h_{1j}}{H_1} &= \frac{h_{1j}\sum_{k=1}^m H_k}{H_1\sum_{k=1}^m H_k} \\ &= \frac{h_{1j}\left(1+\sum_{k=2}^m \frac{H_k}{H_1}\right)}{\sum_{k=1}^m H_k} \\ &= \frac{h_{1j}+\sum_{k=2}^m h_{kj}}{W} \\ &= \frac{\sum_{k=1}^m h_{kj}}{W} \\ &= \frac{w_j}{W}\end{aligned}$$

□

Stock market indices [1] are approximations of market portfolios: the values of these indices are the weighted average price of a large set of stocks, where the weights are proportional to the proportion of the total value of a stock as part of the total value of the whole set of stocks.

You might have also heard of *tracker funds:* these are investments that hold shares in the same proportion as a given index, e.g., FTSE 100 trackers. The previous theorem says roughly that the only risky investments in the portfolio of an investor who assumes axioms A1-A7 must be tracker funds.

There are other types of investment funds, *actively managed funds.* These funds invest money in stocks carefully chosen by spectacularly highly paid fund managers. These funds demand high fees from investors in return for applying their talents in choosing the way in which the fund's assets will be invested. So you are asked to pay large fees to have axiom A6 broken: these fund managers claim to possess knowledge which is not apparent to lesser investors.

There is a ongoing debate on whether these fund managers are worth these high fees.

2. The market price of risk

For any efficient investment A lying on the market line we have

$$r_A - r_B = \frac{r_M - r_B}{\sigma_M}\sigma_A$$

where r_B is the risk-free interest rate and M is the market portfolio. We interpret $\frac{r_M - r_B}{\sigma_M}$ as the market price of risk: this slope measures how much more return investors demand for an increase of one unit in the volatility of their returns.

We want a similar expression for the excess return above the risk-free interest rate for non-efficient portfolios, e.g., individual stocks.

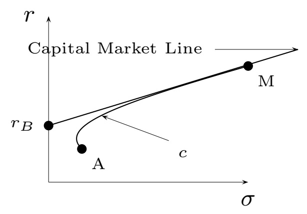
THEOREM 22. *For any portfolio A we have*

$$r_A - r_B = \frac{r_M - r_B}{\sigma_M^2}\operatorname{Covar}(A, M)$$

where r_B is the risk-free interest rate, M is the market portfolio and $\operatorname{Covar}(A, M)$ *is the covariance between the return of A and the return of M.*

[1] e.g., FTS100, the Dow Jones Industrial Index, S&P 500, Nikkei, CAC40, DAX, etc.

Proof. For any $0 \le t \le 1$, let portfolio Π_t consist of an investment of t in A and an investment of $1-t$ in M. The curve c given by $(\sigma_{\Pi_t}, r_{\Pi_t})$ in the σ-r plane joins points A and M.



c intersects the capital market line at M, and the capital market line must be tangent to c at M: Otherwise c would *cross* the capital market line and we would have a portfolio above the capital market line, contradicting the fact that the capital market line is the efficient frontier.

Calculate the slope of c at M: c is given by $t \mapsto \big(\sqrt{t^2\sigma_A^2 + 2t(1-t)\operatorname{Covar}(A,M) + (1-t)^2\sigma_M^2},\ tr_A + (1-t)r_M\big)$ as $(0 \le t \le 1)$. Evaluate the derivative with respect to t

$$\left(\frac{2t\sigma_A^2 + (2-4t)\operatorname{Covar}(A,M) - 2(1-t)\sigma_M^2}{2\sqrt{t^2\sigma_A^2 + 2t(1-t)\operatorname{Covar}(A,M) + (1-t)^2\sigma_M^2}}, r_A - r_M\right)$$

at $t = 0$ to obtain the slope at point M: $\frac{r_A - r_M}{\operatorname{Covar}(A,M) - \sigma_M^2}\sigma_M$. But this slope must be equal to the slope of the capital market line, i.e., $\dfrac{r_A - r_M}{\operatorname{Covar}(A,M) - \sigma_M^2}\sigma_M = \dfrac{r_M - r_B}{\sigma_M}$ and we can rearrange this to obtain $r_A - r_B = \dfrac{r_M - r_B}{\sigma_M^2}\operatorname{Covar}(A,M)$.

□

Definition. The *beta coefficient* of a portfolio A is defined as

$$\beta = \frac{\operatorname{Covar}(A,M)}{\sigma_M^2}.$$

The *security market line* is the linear relation between expected returns r and beta coefficients β given by

$$r = r_B + (r_M - r_B)\beta.$$

We can restate the previous theorem: *for any investment with expected return r and beta coefficient β the point (β, r) lies on the security market line.*