MAS362/MAS462/MAS6053 Financial Mathematics
Problem Sheet 1 Solutions

1. A repayment of x in m months time has present value $x\left(1+\frac{0.05}{12}\right)^{-m}$ now. So we must have

$$100,000 = x\sum_{m=1}^{300}\left(1+\frac{0.05}{12}\right)^{-m} = x\sum_{m=1}^{300}\left(\frac{241}{240}\right)^{m}$$

Solving for x gives

$$x = \frac{100,000}{\sum_{m=1}^{300}\left(\frac{240}{241}\right)^{m}} = \frac{100000}{\frac{240}{241}\frac{1-(240/241)^{300}}{1/241}} = \frac{100000}{240\left(1-\left(\frac{240}{241}\right)^{300}\right)}$$

$$\approx 584.59$$

2. Let x be your investment. In the first account, your final balance will be $1.05x$. In the second, it will be $x(1+0.048/2)^2 = 1.0486x$. In the third, it will be $xe^{0.049} = 1.0502x$. So the third account is best.

3. If the current price p is less than $v_T e^{-rT}$, borrow p for T years at an interest rate of r, and use this cash to buy the asset. At time T, sell the asset for v_T, repay the balance of the loan which is pe^{rT} and pocket the difference

$$v_T - pe^{rT} > v_T - v_T e^{-rT}e^{rT} = 0.$$

If the current price p is greater than $v_T e^{-rT}$, those who own the asset will sell it and lend the proceeds for T years earning an interest rate of r. At time T they will collect the balance of their deposit, now worth pe^{rT}, to buy the asset back and pocket the difference

$$pe^{rT} - v_T > v_T e^{-rT}e^{rT} - v_T = 0.$$

So the only value of p which does not introduce arbitrage opportunities is $p = v_T e^{-rT}$.

4. Payments are due at times (measured in years) $n+1/4$, $n = 0, 1, 2, 3, \ldots$. These payments are of £2.50, so, taking account of the interest rate, the payment at time $n+1/4$ is worth $2.5e^{-0.05(n+1/4)}$. So the total value of all the payments is

$$\sum_{n=0}^{\infty} 2.5e^{-0.05(n+1/4)} = \frac{2.5e^{-1/80}}{1-e^{-0.05}}$$

which is £50.62. Hence this is the value of the bond.

5. We first find the spot interest rates for 6, 12 and 18 months:

$$r_6 = \frac{-\log(98/100)}{0.5} \approx 0.04041,$$

$$r_{12} = \frac{-\log(96/100)}{1} \approx 0.04082,$$

$$r_{18} = \frac{-\log(93.5/100)}{1.5} \approx 0.04480.$$

Call the two year spot interest rate r; it satisfies

$$107 = 4e^{-0.5r_6} + 4e^{-r_{12}} + 4e^{-1.5r_{18}} + 104e^{-2r}.$$

We now solve for r and obtain

$$r = -\frac{1}{2}\log\frac{107 - 4e^{-0.5r_6} + 4e^{-r_{12}} + 4e^{-1.5r_{18}}}{104} \approx 0.04263.$$

The two year discount factor is then $e^{-2r} \approx 0.9182692309$.

6. We first compute the six-month and one-year spot interest rates:

$$Y(0.5) = -\log(97.8/100)/0.5 \approx 4.45\%$$

$$Y(1) = -\log(95.5/100)/1 \approx 4.60\%$$

The forward rate r for the period between 0.5 and 1 years is

$$r \approx \frac{0.046 \times 1 - 0.0445 \times 0.5}{1 - 0.5} \approx .0476 = 4.76\% < 5\%$$

so we enter the agreement as a depositor and

(a) borrow $£1,000,000e^{-Y(0.5)\times 0.5}$ for a year at an interest rate of $Y(1)$,

(b) deposit this money for 6 months at an interest rate of $Y(0.5)$,

(c) after 6 months with withdraw our deposit amounting to $£1,000,000e^{-Y(0.5)\times 0.5}e^{Y(0.5)\times 0.5} = £1,000,000$,

(d) deposit this £1,000,000 for six months at an interest rate of 5%,

(e) after an additional 6 months we withdraw this deposit, whose balance is $1,000,000e^{0.05\times 0.5}$ and

(f) we repay the loan we took in step (1), and whose balance now is $£1,000,000e^{-Y(0.5)\times 0.5}e^{Y(1)\times 1}$,

(g) we pocket the difference

$$1,000,000e^{0.05\times 0.5} - 1,000,000e^{-Y(0.5)\times 0.5}e^{Y(1)\times 1} \approx 1,280.$$