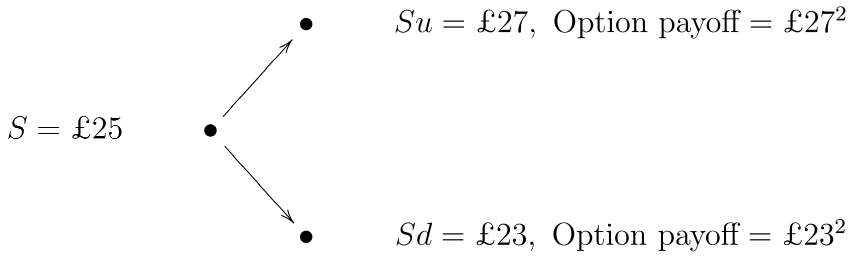
MAS362/MAS462/MAS6051 Financial Mathematics
Problem Sheet 3 Solutions

1. Modeling this as a binomial tree, we have $u = 27/25$, $d = 23/25$, $r = 0.1$ and the risk-neutral probability of asset price going up is $q = (e^{r\times t} - d)/(u - d) = 0.6050$.

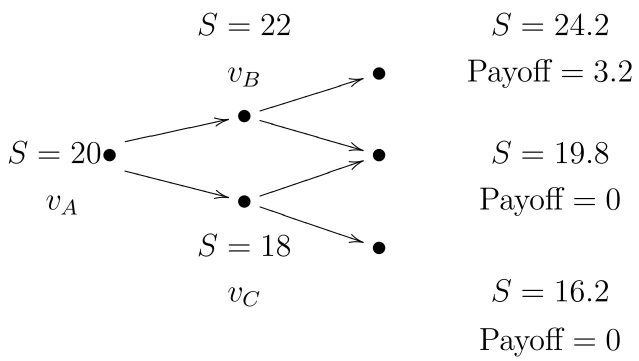


The present value of the expected payoff in a risk-neutral world is

$$e^{-0.1\times 1/6}\left(27^2 \times 0.605 + 23^2 \times 0.3950\right) = 639.26.$$

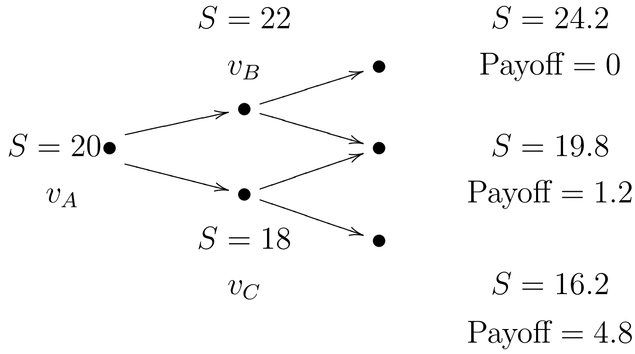
2. The risk-neutral probability is $q = \frac{e^{0.01} - 0.9}{0.2} = 0.5503$.

Consider the following binomial tree describing the share price and call option value over the periods:



The value of the derivative after one "up" share price movement is $v_B = e^{-0.01}(q \times 3.2 + (1-q) \times 0) \approx 1.743$, and after a "down" movement is $v_C = e^{-0.01}(q \times 0 + (1-q) \times 0) = 0$ and the value at time zero is $v_A \approx e^{-0.01}(q \times 1.743 + (1-q) \times 0) \approx 0.9496$.

For the put option use the binomial tree



We now calculate $v_B = e^{-0.01}(q \times 0 + (1-q) \times 1.2) \approx 0.5343$ $v_C = e^{-0.01}(q \times 1.2 + (1-q) \times 4.8) \approx 2.7909$, and $v_A = e^{-0.01}(q \times v_B + (1-q) \times v_C) \approx 1.5337$

3. Revisiting the calculations in the previous problem we verify that the call option should not be exercised early: $1.743 > 1$, and $0.9496 > 0$. Hence the value of the American call option is the same as the European call option.

   Consider now the American put option. Consider the "down" node in the tree: the payoff from an immediate exercise would be 3. The discounted expected (in the risk neutral universe) payoff from waiting would be $e^{-0.01}(1.2q+4.8(1-q)) = 2.791$. So it is optimal to **exercise now**, with payoff (and hence current value) 3.

   Now consider the "up" node: the payoff from an immediate exercise would be zero. The discounted expected (in the risk neutral universe) payoff from waiting would be $e^{-0.01}(0q + 1.2(1-q)) = 0.5343$. So it is optimal to **wait**, with the current value 0.5343.

   At the initial node, the discounted risk neutral expectation of the value in one month's time will be $e^{-0.01}(0.5353q + 3(1-q)) = 1.627$. We could exercise immediately, with payoff 1, but this would not be optimal. So the value of the American put is 1.627 (pounds).

   The only case where an early exercise is optimal is when the first step is downwards.

4. (a) $E(X) = 27/36 \times (-1) + 9/36 \times 1 = -1/2$, $E(X^2) = 27/36 \times (-1)^2 + 9/36 \times 1^2 = 1$, and $V(X) = E(X^2) - E(X)^2 = 3/4$.

   $E(Y) = 12/36 \times 1 + 24/36 \times 2 = 60/36 = 5/3$, $E(Y^2) = 12/36 \times 1 + 24/36 \times 4 = 108/36 = 3$, and $V(Y) = E(Y^2) - E(Y)^2 = 2/9$.

   (b) $E(XY) = -5/36 - 44/36 + 7/36 + 4/36 = -19/18$, $Cov(X,Y) = E(XY) - E(X)E(Y) = -2/9$. $\rho(X,Y) = Cov(X,Y)/\sqrt{(}V(X)V(Y)) = -2\sqrt{6}/9$.

   (c) $E(X/3 + 2Y/3 + 1) = E(X)/3 + 2E(Y)/3 + 1 = -1/6 + 10/9 + 1$.
   $V(X/3+2Y/3+1) = V(X/3+2Y/3) = V(X/3)+V(2Y/3)+2Cov(X/3,2Y/3) = V(X)/9 + 4/9 \times V(Y) + 4/9Cov(X,Y) = 3/36 + 8/81 - 8/81 = 3/36$.

5. The event $X < a$ is the same as the event $\log X < \log a$, and this has probability $\Phi((\log a - \mu)/\sigma)$ where Φ is the distribution function of a standard normal random variable.

6. $X_1 + \cdots + X_n$ is normally distributed with mean $\mu_1 + \cdots + \mu_n$ and variance $\sigma_1^2 + \ldots \sigma_n^2$.

7.
$$\frac{\partial af_1 + bf_2}{\partial t} + rS\frac{\partial af_1 + bf_2}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 af_1 + bf_2}{\partial S^2} =$$

$$a\frac{\partial f_1}{\partial t} + arS\frac{\partial f_1}{\partial S} + \frac{1}{2}a\sigma^2 S^2\frac{\partial^2 f_1}{\partial S^2} + b\frac{\partial f_2}{\partial t} + brS\frac{\partial f_2}{\partial S} + \frac{1}{2}b\sigma^2 S^2\frac{\partial^2 f_2}{\partial S^2} = arf_1 + brf_2 = r(af_1 + bf_2).$$

$$\frac{\partial kS}{\partial t} + rS\frac{\partial kS}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 kS}{\partial S^2} = 0 + krS + 0 = rkS$$

$$\frac{\partial ke^{rt}}{\partial t} + rS\frac{\partial ke^{rt}}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 ke^{rt}}{\partial S^2} = 0 + rke^{rt} + 0 + 0 = rke^{rt}$$