MAS362/MAS462/MAS6051 Financial Mathematics
Problem Sheet 4 Solutions

1. Apply Ito's Lemma to the process X_t given by $dX = 0dt + dB$ (i.e., $X = B$) and the function $G(x,t) = f(t)x$. $\dfrac{\partial G}{\partial x} = f(t)$, $\dfrac{\partial^2 G}{\partial x^2} = 0$ and $\dfrac{\partial G}{\partial t} = f'(t)x$ and we obtain

$$d(f(t)B_t) = \left(0 + f'(t)X + \frac{1}{2}0\right)dt + f(t)dB = f'(t)Bdt + f(t)dB.$$

   This is another way of saying that

$$f(t)B_t = \int_0^t f'(s)B_s ds + \int_0^t f(s)dB_s.$$

   The second assertion follows by rearranging terms.

2. The parameters are $S_0 = 10$, $\mu = 0.15$, $\sigma = 0.2$. Now $\log S_T \sim N(\log S_0 + (\mu - \sigma^2/2)T, \sigma^2 T)$ and $\log S_1 \sim N(\log 10 + (0.15 - 0.2^2/2), 0.2^2) = N(\log 10 + 0.13, 0.04)$. So the required probability is $\Phi((\log 8 - (\log 10 + 0.13))/0.2) \approx 3.87\%$.

3. We have

$$\frac{\partial f}{\partial S} = -1$$
$$\frac{\partial^2 f}{\partial S^2} = 0$$
$$\frac{\partial f}{\partial t} = rXe^{-r(T-t)}$$

   and so
$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rXe^{-r(T-t)} - rS = rf.$$

   $p(S_T, T) = \max\{X - S_T, 0\} \geq (X - S_T) = f(S_T, T)$.

   Since $f(S,t)$ satisfies the Black-Scholes partial differential equation, it is equal to the price of a European style option with a single payoff at time T of an amount equal to $f(S_T, T)$.

   The payoff $p(S_T, T)$ of the put option is always at least as much as the payoff the option above.
   so at any time $t < T$, the value of the put option is at least as valuable as the the option above,
   i.e.,
$$p(S,t) \geq f(S,t) = S - Xe^{-r(T-t)}.$$

4.

$$\frac{\partial f}{\partial t} = \frac{4r+\sigma^2}{8} f(S,t)$$
$$\frac{\partial f}{\partial S} = \frac{1}{2} S^{-1/2} e^{\frac{4r+\sigma^2}{8}t}$$
$$\frac{\partial^2 f}{\partial S^2} = -\frac{1}{4} S^{-3/2} e^{\frac{4r+\sigma^2}{8}t}$$

and

$$\begin{aligned}\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} &= \frac{4r+\sigma^2}{8} f(S,t) + \frac{1}{2} r f(S,t) - \frac{1}{8}\sigma^2 f(S,t) \\ &= r f(S,t)\end{aligned}$$

Consider now the function $v(S,t) = e^{\frac{4r+\sigma^2}{8}(t-T)}\sqrt{S}$; this function is a multiple of $f(S,t)$ and since the Black-Scholes PDE is linear, we conclude that $v(S,t)$ is a also a solution of this PDE. The function v also satisfies the boundary condition $v(S,T) = \sqrt{S}$ and $v(0,t) = 0$ for all $0 \le t \le T$ and so v gives the price of the option.

5. Both c_1 and c_2 satisfy the Black-Scholes PDE, and since it is linear, so does any linear combination of c_1 and c_2.

   Write $v(S,t) = c_1(S,t) - c_2(S,t)$. Now $v(S_T,t) = c_1(S_T,t) - c_2(S_T,t)$ and v is the price of a European option whose payoff at time T is $c_1(S_T,T) - c_2(S_T,T)$. Since

   $$0 \le c_1(S_T,T) - c_2(S_T,T) = \max\{S_T - X_1, 0\} - \max\{S_T - X_2, 0\} \le X_2 - X_1,$$

   we have $v(S,t) \ge 0$ (one can only gain from holding this option) and $v(S,t) \le (X_2 - X_1)e^{-r(T-t)}$ (the value of the option cannot exceed the present value of the maximal payoff.)

6. The risk neutral valuation principle states that in valuing a derivative producing a payoff at some time in the future, which is a function of the underlying asset price at that time, one may assume that: (a) the value of the derivative is the expected value of the present value of the payoff, and
   (b) the underlying asset has an expected return equal to the risk-free return.

7. When $T_1 \le t \le T_2$, S_{T_1} is known and to find the value of the option we just use the Black-Scholes pricing formula with $X = S_{T_1}$.

   Consider now the value at $t = T_1$. We compute:

   $$d_1 = \frac{\log(S_{T_1}/S_{T_1}) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = \frac{r + \sigma^2/2}{\sigma}\sqrt{T_2 - T_1}$$

and

$$d_2 = \frac{\log(S_{T_1}/S_{T_1}) + (r - \sigma^2/2)\,(T-t)}{\sigma\sqrt{T-t}} = \frac{r-\sigma^2/2}{\sigma}\sqrt{T_2 - T_1}$$

and we notice that d_1 and d_2 do not depend on the stock price in any way. Now the price of the option at time $t = T_1$ is

$$S_{T_1}\Phi(d_1) - S_{T_1}e^{-r(T_2-T_1)}\Phi(d_2) = \left(\Phi(d_1) - e^{-r(T_2-T_1)}\Phi(d_2)\right)S_{T_1}.$$

Now consider a portfolio consisting of $\left(\Phi(d_1) - e^{-r(T_2-T_1)}\Phi(d_2)\right)$ shares. At time $t = T_1$ this portfolio will have exactly the same value as our forward start option, and this implies that the value of the option for any $0 \le t \le T_1$ must be the same as the value of the portfolio at that time, i.e.,

$$\left(\Phi(d_1) - e^{-r(T_2-T_1)}\Phi(d_2)\right)S_t.$$