MAS362/MAS462/MAS6051 Financial Mathematics
Problem Sheet 4

1. Let $f(t)$ be a differentiable function.

   (a) Describe the process $f(t)B_t$.

   (b) Deduce that
   $$\int_0^t f(s)dB_s = f(t)B_t - \int_0^t f'(s)B_s ds.$$

2. Consider a stock currently trading at £10, with expected annual return of 15% and annual volatility of 0.2. Under our standard assumption about the evolution of stock prices, what is the probability that the price of the stock in one year's time will be below £8?

3. Let S denote the price of a stock paying no dividends, and consider a European put option on this stock with strike price X and expiring in T years. Let $p(S,t)$ denote the price of this option at time $0 \le t \le T$. Assume all interest rates are constant and equal to r. Again we assume that S follows the process
   $$dS = \mu S dt + \sigma S dB$$
   for constants μ, σ and where B is a Brownian motion.

   (a) Show that the function
   $$f(S,t) = Xe^{-r(T-t)} - S$$
   satisfies the Black-Scholes partial differential equation
   $$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

   (b) Let S_T denote the stock price at time T. Explain why $p(S_T,T) \ge f(S_T,T)$.

   (c) Explain why the results in (a) and (b) imply that
   $$p(S,t) \ge Xe^{-r(T-t)} - S.$$

4. (a) Show that $f(S,t) = e^{\frac{4r+\sigma^2}{8}t}\sqrt{S}$ is a solution of the Black-Scholes PDE.

   (b) Consider a derivative on certain stock (whose price S, as always, follows the process $dS = S\mu dt + S\sigma dB$) which provides a single payoff at time $T > 0$ amounting to $\sqrt{S_T}$. Find the value of the derivative at time $0 \le t \le T$.

5. Let $c_1(S,t)$ and $c_2(S,t)$ be the prices at time t of two European call options on the same non-dividend paying stock with price S, with same expiration T and with strike prices X_1 and X_2, respectively. Assume that $X_1 < X_2$.

(a) Explain why $c_1 - c_2$ is a solution of the Black-Scholes PDE.

(b) By considering $c_1(S_T, T) - c_2(S_T, T)$ deduce that

$$0 \leq c_1(S_t, t) - c_2(S_t, t) \leq (X_2 - X_1)e^{-r(T-t)}.$$

6. Explain the principle of risk neutral valuation.

7. Consider an option on a stock which gives its holder at time T_1 a European call option on the stock whose strike price is S_{T_1}, the stock price at time T_1, and which expires at time $T_2 > T_1$. These options are known as *forward start options.* Find a formula for the value of this option at any time $0 \leq t \leq T_2$. (Hint: distinguish between the cases $0 \leq t \leq T_1$ and $T_1 \leq t \leq T_2$. Use the Black-Scholes pricing formula to find the value of the option at time $t = T_1$.)